October 17 MATH 1113 sec. 52 Fall 2018

Section 5.5: Solving Exponential and Logarithmic Equations

Base-Exponent Equality For any $a > 0$ with $a \neq 1$, and for any real numbers x and y

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

Logarithm Equality For and $a > 0$ with $a \neq 1$, and for any positive numbers x and y

$$\log_a x = \log_a y \quad \text{if and only if} \quad x = y.$$

Inverse Function For any $a > 0$ with $a \neq 1$

$$a^{\log_a x} = x \quad \text{for every} \quad x > 0$$

$$\log_a(a^x) = x \quad \text{for every real} \quad x.$$

Log Equations & Verifying Answers

Double checking answers is always recommended. When dealing with functions whose domains are restricted, **answer verification** is critical.

Use properties of logarithms to solve the equation
$2\log_5(x) = \log_5(x+10) + 1$

$\log_5(5) = 1$

$\log_5(x^2) = \log_5(x+10) + \log_5(5)$

$\log_5(x^2) = \log_5\left(5(x+10)\right)$

$x^2 = 5(x+10)$

$x^2 - 5x - 50 = 0 \Rightarrow (x-10)(x+5) = 0$

$x = 10$ or $x = -5$

Both solve the quadratic equation. We'll check to see if they solve the log equation

$$2\log_5(x) = \log_5(x+10) + 1 = \log_5(x+10) + \log_5(5)$$

Check 10: LHS $2\log_5(10) = \log_5(10^2) = \log_5(100)$

RHS $\log_5(10+10) + \log_5(5) = \log_5(20) + \log_5(5)$

$= \log_5(20\cdot 5)$

$= \log_5(100)$

10 is a solution.

Check -5: LHS $2\log_5(-5)$ undefined

-5 is not a solution

There is one solution, 10.

Question

Solve the equation $\log_6 x + \log_6(x-1) = 1$. (Hint: $\log_6 6 = 1$.)

(a) $x = 3$ or $x = -2$

(b) $x = 2$ or $x = -3$

(c) $x = 3$

(d) $x = 2$

(e) $x = 0$ or $x = 1$

$\log_6 (x(x-1)) = \log_6 6$

$x(x-1) = 6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$

not a solution since $\log_6(-2)$ isn't defined.

Combining Skills

Find all solutions of the equation[1]

$$\frac{e^x + e^{-x}}{2} = 2$$

$e^x + e^{-x} = 4$

We'll multiply by e^x

$e^x(e^x + e^{-x}) = 4e^x$

$(e^x)^2 + e^{-x}e^x = 4e^x$

$e^{-x}e^x = e^0 = 1$

$(e^x)^2 + 1 = 4e^x \Rightarrow (e^x)^2 - 4e^x + 1 = 0$

[1]The function $f(x) = \frac{e^x + e^{-x}}{2}$ is a well known function called the *hyperbolic cosine*.

Letting $u = e^x$, we have a quadratic equation

$$u^2 - 4u + 1 = 0$$ Let's complete the square

$$u^2 - 4u + 4 - 4 + 1 = 0$$

$$(u-2)^2 - 3 = 0$$

$$(u-2)^2 = 3$$

$$u - 2 = \pm\sqrt{3} \quad \Rightarrow \quad u = 2 \pm \sqrt{3}$$

There are two possible solutions

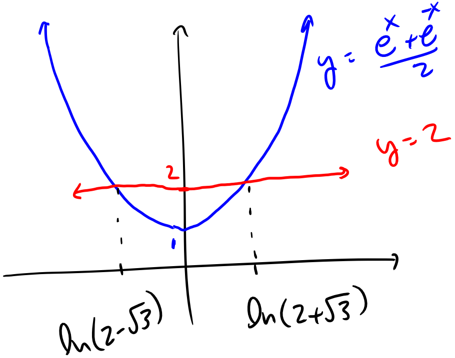
$$e^x = 2 + \sqrt{3} \quad \text{or} \quad e^x = 2 - \sqrt{3}$$

Both are positive, so both are valid.

The two solutions come from

$$\ln(e^x) = \ln(2+\sqrt{3}) \quad \text{and} \quad \ln(e^x) = \ln(2-\sqrt{3})$$

$$x = \ln(2+\sqrt{3}) \qquad\qquad x = \ln(2-\sqrt{3})$$



Applications

When a quantity Q changes in time according to the model $Q(t) = Q_0e^{kt}$, it is said to experience *exponential growth* ($k > 0$) or *exponential decay* ($k < 0$). Here, Q_0 is the constant initial quantity $Q(0)$, and k is a constant. Examples of such phenomena include

- population changes (over small time frame),
- continuously compounded interest (on deposit or loan amount),
- substances subject to radio active decay

Generally, the model for exponential decay is written $Q(t) = Q_0e^{-kt}$, and for growth it is written as $Q(t) = Q_0e^{kt}$ so that it is always assumed that $k > 0$.

Example

The ^{44}Ti titanium isotope decays to ^{44}Ca, a stable calcium isotope. The mass Q is subject to exponential decay

$$Q(t) = Q_0 e^{-kt} \quad \text{for } t \text{ in years, and some} \quad k > 0.$$

If the half life (amount of time for the mass to reduce by 50%) is 60 years, determine the value of k.

We're given that $Q(60) = \frac{1}{2} Q(0)$

$$Q(0) = Q_0 e^{-k \cdot 0} = Q_0 \cdot 1 = Q_0$$

$$Q(60) = Q_0 e^{-k \cdot 60} = Q_0 e^{-60k}$$

Setting $Q(60)$ equal to $\frac{1}{2} Q(0)$

$$Q(60) = \frac{1}{2} Q(0)$$

$$Q_0 e^{-60k} = \frac{1}{2} Q_0$$

Divide by Q_0, solve for k.

$$e^{-60k} = \frac{1}{2}$$

Take the natural log

$$\ln\left(e^{-60k}\right) = \ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right)$$

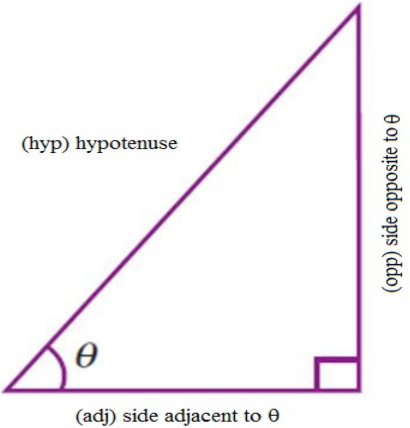
$$-60k = -\ln 2$$

$$k = \frac{-\ln 2}{-60} = \frac{\ln 2}{60}$$

So $k = \frac{\ln 2}{60}$ and $Q(t) = Q_0 e^{-\frac{\ln 2}{60} t}$

Sections 6.1 & 6.2: Trigonometric Functions of Acute Angles

In this section, we are going to define six new functions called **trigonometric functions.** We begin with an acute angle θ in a right triangle with the sides whose lengths are labeled:



Sine, Cosine, and Tangent

For the acute angle θ, we define the three numbers as follows

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \text{read as "sine theta"}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \text{read as "cosine theta"}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}, \qquad \text{read as "tangent theta"}$$

Note that these are numbers, ratios of side lengths, and have no units.

It may be convenient to enclose the argument of a trig function in parentheses. That is,

$$\sin\theta = \sin(\theta).$$

Cosecant, Secant, and Cotangent

The remaining three trigonometric functions are the reciprocals of the first three

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}, \qquad \text{read as "cosecant theta"}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}, \qquad \text{read as "secant theta"}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\tan\theta}, \qquad \text{read as "cotangent theta"}$$

A Word on Notation

The trigonometric ratios define functions:

$$\text{input angle number} \rightarrow \text{output ratio number.}$$

From the definitions, we see that

$$\csc\theta = \frac{1}{\sin\theta}.$$

Functions have arguments. It is NOT acceptable to write the above relationship as

$$\csc = \frac{1}{\sin}.$$