Oct. 17 Math 1190 sec. 51 Fall 2016

Section 4.2: Maximum and Minimum Values; Critical Numbers

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Some authors call critical numbers *critical points*.

Example

Find all critical numbers of the function

(b) $g(t) = t^{1/5}(12 - t)$

The domain is $(-\infty, \infty)$.

$g(t) = 12t^{1/5} - t^{6/5}$

$g'(t) = 12\left(\frac{1}{5}t^{1/5-1}\right) - \frac{6}{5}t^{6/5-1}$

$= \frac{12}{5}t^{-4/5} - \frac{6}{5}t^{1/5}$

$= \frac{12}{5t^{4/5}} - \frac{6t^{1/5}}{5} \cdot \frac{t^{4/5}}{t^{4/5}}$

$$g'(t) = \frac{12}{5t^{4/5}} - \frac{6t}{5t^{4/5}}$$

$$= \frac{12-6t}{5t^{4/5}}$$

Find where $g'(t) = 0$:

$$12 - 6t = 0$$

$$12 = 6t \Rightarrow t = \frac{12}{6} = 2$$

Find where $g'(t)$ is undefined:

$$5t^{4/5} = 0 \Rightarrow t = 0.$$

The critical numbers are 2 and 0.

we could say that the set of critical numbers is $\{2, 0\}$.

Question

Find all critical numbers of the function h. Note that the domain of h is $(-\infty, 0) \bigcup (0, \infty)$.

$$h(x) = x + \frac{1}{x}$$

$$h'(x) = \frac{x^2 - 1}{x^2}$$

$h'(x) = 0$ if $x^2 - 1 = 0$

$x^2 = 1 \Rightarrow x = \pm 1$

This is all since $x = 0$ is not in the domain.

(a) $\{1, -1, 0\}$

(b) $\{1\}$

(c) $\{1, -1\}$

(d) $\{-1\}$

Extreme Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(c)$ and f attains an absolute minimum value $f(d)$ for some numbers c and d in $[a, b]$.

Recall that this says that if (1) f is continuous, and (2) the interval is closed, we're guaranteed to have f actually have an absolute maximum and an absolute minimum on the interval.

In practice, we look for these by considering critical points inside the interval and considering the endpoints.

Example

Find the absolute maximum and absolute minimum values of the function on the closed interval.

(a) $g(t) = t^{1/5}(12 - t)$, on $[-1, 1]$

Note that g is continuous on $(-\infty, \infty)$, so it's continuous on the closed interval $[-1, 1]$.

We'll look for critical numbers in $(-1, 1)$. Then we'll compare function values at the critical numbers and the end points.

We found critical numbers 2 and 0. Zero is inside the interval. With the end points,

we compare

end → $g(-1) = (-1)^{1/3}(12-(-1)) = -1(13) = -13$ ← minimum

critical no. → $g(0) = (0)^{1/5}(12-0) = 0$

$g(1) = (1)^{1/5}(12-1) = 1 \cdot 11 = 11$

end ↗ ↖ maximum

The absolute maximum value is $11 = g(1)$. The absolute minimum value is $-13 = g(-1)$.

(b) $f(x) = xe^x$, on $[-3, 1]$

f is continuous on $(-\infty, \infty)$, so it's continuous on $[-3, 1]$. Let's look for critical numbers inside $(-3, 1)$.

$$f'(x) = 1 \cdot e^x + x \cdot e^x$$
$$= e^x(1+x).$$

When is $f'(x)$ undefined? Never

When is $f'(x) = 0$? $\quad e^x(1+x) = 0$

$$\Rightarrow e^x = 0 \quad \text{or} \quad 1+x=0$$

$e^x > 0$ for all x, $\quad 1+x=0 \Rightarrow x=-1$

We have one critical number in $(-3,1)$, namely -1.

We'll compare

$$f(-3) = -3e^{-3} = \frac{-3}{e^3}$$

$$f(-1) = -1e^{-1} = \frac{-1}{e}$$

$$f(1) = 1 \cdot e^1 = e$$ ← obviously the largest

Notice that $3 < e^2$

$$-3 > -e^2$$

$$\frac{-3}{e^3} > \frac{-e^2}{e^3} = \frac{-1}{e}$$

So $\frac{-1}{e}$ is the smallest number in the list.

It turns out that

$\frac{-1}{e} \approx -0.3679$ and

$$\frac{-3}{e^3} \approx -0.1494$$

The absolute maximum value is $e = f(1)$, and

the absolute minimum value is $\frac{-1}{e} = f(-1)$.

Question

Find the absolute maximum and absolute minimum values of the function on the closed interval.

critical numbers:

3 and -3

not in interval

$f(x) = 1 + 27x - x^3, \quad \text{on} \quad [0, 4]$

$f'(x) = 27 - 3x^2 = 3(9 - x^2)$

(a) Minimum value is 1, maximum value is 55

$f(0) = 1$

(b) Minimum value is 1, maximum value is 35

$f(3) = 55$

$f(4) = 45$

(c) Minimum value is -53, maximum value is 55

(d) Minimum value is -53, maximum value is 35

Section 4.3: The Mean Value Theorem

Rolle's Theorem: Let f be a function that is

i continuous on the closed interval $[a, b]$,

ii differentiable on the open interval (a, b), and

iii such that $f(a) = f(b)$.

Then there exists a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem (MVT) is arguably the most significant theorem in calculus. This is even accounting for a theorem we'll discuss later called the *Fundamental Theorem of Calculus*.

Example

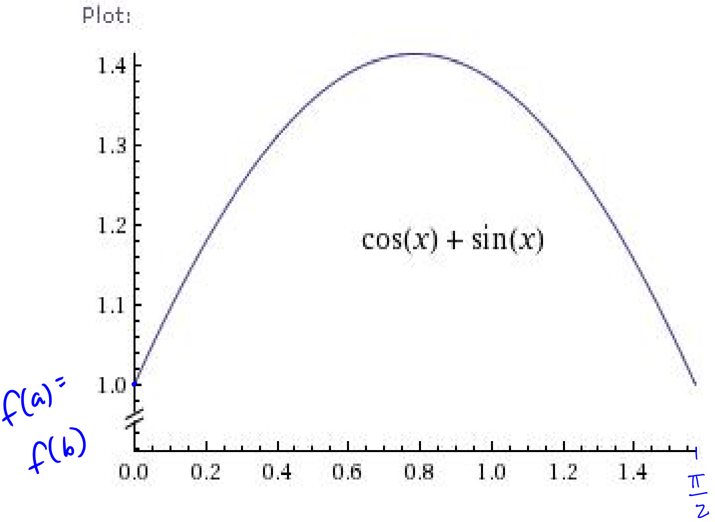
Show that the function $f(\theta) = \cos\theta + \sin\theta$ has at least one point c in $\left[0, \frac{\pi}{2}\right]$ such that $f'(c) = 0$.

Note: (i) f is continuous everywhere, so it's continuous on the closed interval $[0, \pi/2]$.

(ii) f is differentiable everywhere, hence it's differentiable on $(0, \pi/2)$.

(iii) $\left.\begin{array}{l} f(0) = \cos 0 + \sin 0 = 1 + 0 = 1 \\ f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 0 + 1 = 1 \end{array}\right\} \Rightarrow f(a) = f(b)$

By Rolle's thm, there exists c in $(0, \pi/2)$ such that $f'(c) = 0$.



Figure