Oct. 17 Math 1190 sec. 52 Fall 2016

Section 4.2: Maximum and Minimum Values; Critical Numbers

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Some authors call critical numbers *critical points*.

Example

Find all of the critical numbers of the function.

(a) $f(x) = x^4 - 2x^2 + 5$

Note the domain is $(-\infty, \infty)$.

$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$.

When is $f'(x) = 0$? $4x(x-1)(x+1) = 0$

$\Rightarrow 4x = 0$ or $x - 1 = 0$ or $x + 1 = 0$

$\Rightarrow x = 0$ or $x = 1$ or $x = -1$

When is $f'(x)$ undefined? Nowhere $f'(x)$ is defined at all real numbers.

f has 3 critical numbers, 0, 1, and -1.

We could say the set of critical numbers is $\{0, 1, -1\}$.

Example

Find all critical numbers of the function

(b) $g(t) = t^{1/5}(12-t)$ The domain is $(-\infty, \infty)$.

$$g(t) = 12\, t^{1/5} - t^{6/5}$$

$$g'(t) = 12\left(\frac{1}{5}\, t^{\frac{1}{5}-1}\right) - \frac{6}{5}\, t^{\frac{6}{5}-1}$$

$$= \frac{12}{5}\, t^{-4/5} - \frac{6}{5}\, t^{1/5}$$

$$= \frac{12}{5t^{4/5}} - \frac{6t^{1/5}}{5}\,\frac{t^{4/5}}{t^{4/5}}$$

$$= \frac{12}{5t^{4/5}} - \frac{6t}{5t^{4/5}} = \frac{12-6t}{5t^{4/5}}$$

$$g'(t) = \frac{12-6t}{5t^{4/5}}$$

When is $g'(t)=0$? when $12-6t=0 \Rightarrow 12=6t$

$\Rightarrow t=2$

When is $g'(t)$ undefined? when $5t^{4/5}=0 \Rightarrow t=0$.

The critical numbers of g are 2 and 0.

In set notation $\{0,2\}$.

Question

Find all critical numbers of the function h. Note that the domain of h is $(-\infty,0)\bigcup(0,\infty)$.

$h(x) = x + \frac{1}{x}$ $= x + x^{-1}$

$h'(x) = \frac{x^2-1}{x^2}$

$h'(x) = 1 + (-1)x^{-2}$

$= 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$

(a) $\{1, -1, 0\}$

$h'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

(b) $\{1\}$

$h'(x)$ is undefined if $x=0$, but zero isn't in the domain.

(c) $\{1, -1\}$

(d) $\{-1\}$

Extreme Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(c)$ and f attains an absolute minimum value $f(d)$ for some numbers c and d in $[a, b]$.

Recall that this says that if (1) f is continuous, and (2) the interval is closed, we're guaranteed to have f actually have an absolute maximum and an absolute minimum on the interval.

In practice, we look for these by considering critical points inside the interval and considering the endpoints.

Example

Find the absolute maximum and absolute minimum values of the function on the closed interval.

(a) $g(t) = t^{1/5}(12-t)$, on $[-1,1]$

g is continuous on $(-\infty,\infty)$, so it's continuous on the closed interval $[-1,1]$. We found that g has two critical numbers, 0 and 2. Only zero is inside $[-1,1]$. We'll compare the function values at the critical numbers in $(-1,1)$ and at the end points.

$$g(t) = t^{1/5}(12-t)$$

end point

$g(-1) = (-1)^{1/5}(12-(-1)) = -1 \cdot 13 = -13$ smallest

critical number

$g(0) = 0^{1/5}(12-0) = 0$

$g(1) = 1^{1/5}(12-1) = 1 \cdot 11 = 11$ biggest

end point

The absolute minimum value is $-13 = g(-1)$. The absolute maximum value is $11 = g(1)$.

(b) $f(x) = xe^x$, on $[-3, 1]$

f is continuous on $(-\infty, \infty)$, hence on the closed interval $[-3, 1]$. Let's find critical numbers.

$$f'(x) = 1 \cdot e^x + x e^x = e^x(1+x)$$

When is $f'(x) = 0$? $e^x(1+x) = 0 \Rightarrow e^x = 0$ or $1+x=0$

$e^x > 0$ so no solutions there

$1+x=0 \Rightarrow x=-1$

When is $f'(x)$ undefined? Never

f has one critical number -1, and it's in the interval $[-3,1]$.

Compare f at the ends and critical numbers.

$$f(-3) = -3e^{-3} = \frac{-3}{e^3}$$

$$f(-1) = -1e^{-1} = \frac{-1}{e}$$

$$f(1) = 1 \cdot e^{1} = e$$

← obviously the biggest

Note $\quad 3 < e^2 \quad \Rightarrow \quad -3 > -e^2$

$$\Rightarrow \quad \frac{-3}{e^3} > \frac{-e^2}{e^3} = \frac{-1}{e}$$

So $\frac{-1}{e}$ is the smallest number in our list.

Turns out $\quad \frac{-3}{e^3} \approx -0.1494$ and $\frac{-1}{e} \approx -0.3679$

The absolute maximum value is $f(1) = e$, and the absolute minimum is $\frac{-1}{e} = f(-1)$.

Question

Find the absolute maximum and absolute minimum values of the function on the closed interval.

Crit #'s 3, -3

↑ ignore outside of [0, 4]

$f(x) = 1 + 27x - x^3, \quad \text{on} \quad [0, 4]$

$f'(x) = 27 - 3x^2 = 3(9 - x^2)$

(a) Minimum value is 1, maximum value is 55

(b) Minimum value is 1, maximum value is 35

(c) Minimum value is -53, maximum value is 55

(d) Minimum value is -53, maximum value is 35

$f(0) = 1$

$f(3) = 55$

$f(4) = 45$

Section 4.3: The Mean Value Theorem

Rolle's Theorem: Let f be a function that is

i continuous on the closed interval $[a, b]$,

ii differentiable on the open interval (a, b), and

iii such that $f(a) = f(b)$.

Then there exists a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem (MVT) is arguably the most significant theorem in calculus. This is even accounting for a theorem we'll discuss later called the *Fundamental Theorem of Calculus*.

Example

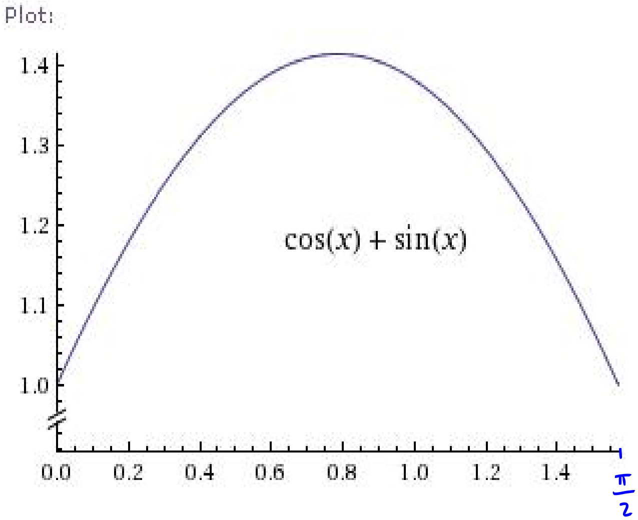
Show that the function $f(\theta) = \cos\theta + \sin\theta$ has at least one point c in $\left[0, \frac{\pi}{2}\right]$ such that $f'(c) = 0$.

Note: (i) f is continuous on $(-\infty, \infty)$, hence on $\left[0, \frac{\pi}{2}\right]$.

(ii) f is differentiable on $(-\infty, \infty)$, hence on $\left(0, \frac{\pi}{2}\right)$.

(iii)
$$\left.\begin{aligned} f(0) &= \cos 0 + \sin 0 = 1 + 0 = 1 \\ f\left(\tfrac{\pi}{2}\right) &= \cos\tfrac{\pi}{2} + \sin\tfrac{\pi}{2} = 0 + 1 = 1 \end{aligned}\right\} \; f(a) = f(b)$$

By Rolle's thm, there must be a number c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.



Figure