October 17 Math 2306 sec. 53 Fall 2018

Section 11: Linear Mechanical Equations

Simple Harmonic Motion: In the absence of any damping or external driving force, we determined the displacement x from equilibrium of an object suspended from a spring according to Hooke's law:

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The value

$$\omega^2 = \frac{k}{m}$$

where k is the spring constant and m the mass of the suspended object.

The equation of motion

The solution to the IVP $x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$ is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.**

We took the sign convention that the direction up is positive ($x > 0$) and down is negative ($x < 0$).

Free Damped Motion

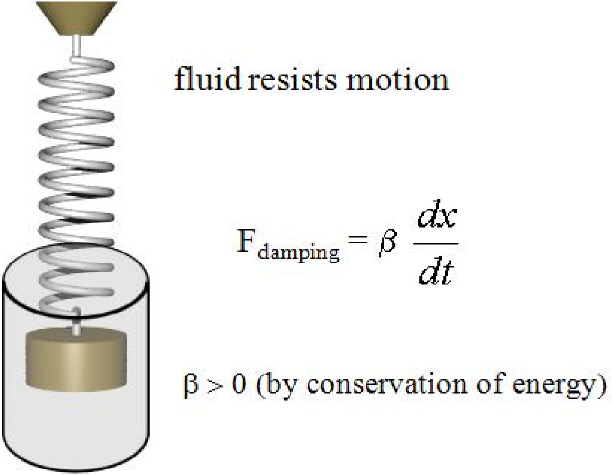

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

$$\text{Total Force} = \text{Force of spring} + \text{Force of damping}$$

$mx'' + \beta x' + kx = 0$

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

2 distinct real roots

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

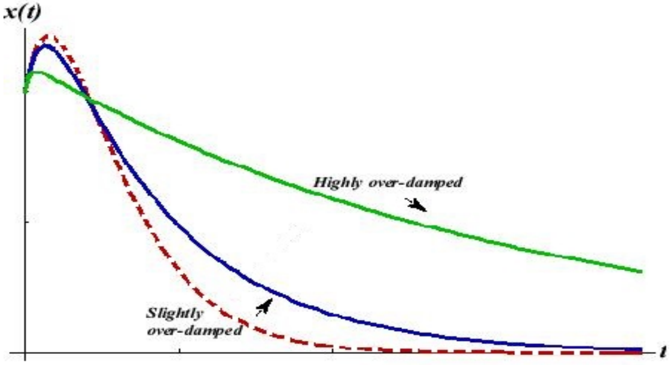

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow. (The red curve is one repeated root case and is shown for reference only.)

Case 2: $\lambda^2 = \omega^2$ Critically Damped

1 repeated real root case

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

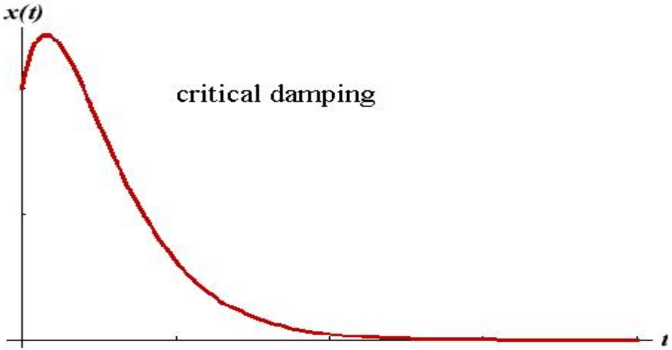

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

Complex conjugate case

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

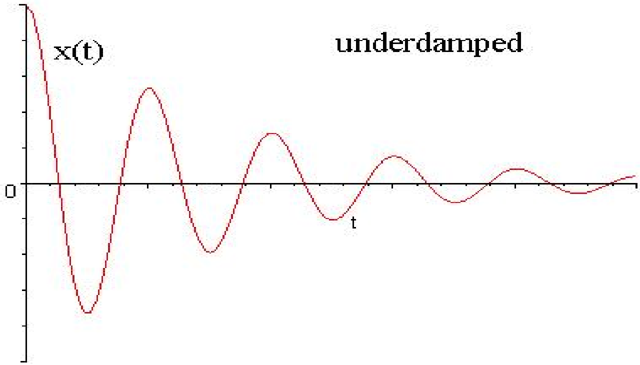

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

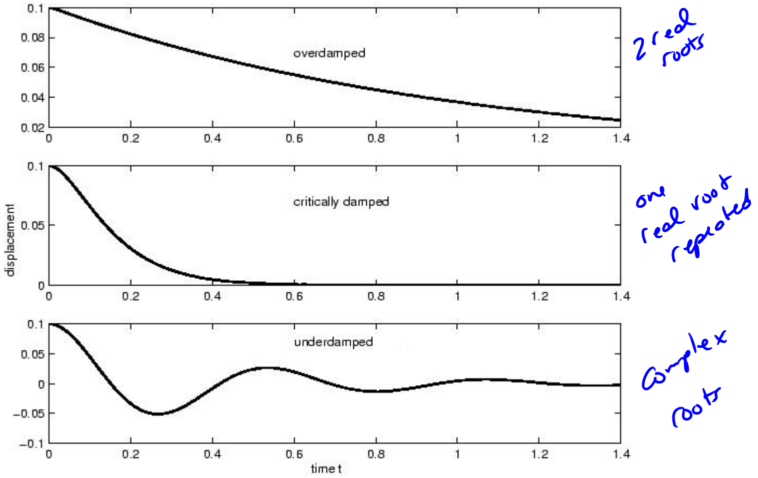

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The ODE looks like

$$mx'' + \beta x' + kx = 0$$

Given $m = 2$ kg , $k = 12$ N/m

10 times instantaneous velocity is $10x'$

$$\beta = 10$$

$$2x'' + 10x' + 12x = 0$$

The characteristic equation is

$$2r^2 + 10r + 12 = 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0 \implies r = -2 \text{ or } r = -3$$

There are 2 distinct real roots, the system is over damped.

Note $2\lambda = \beta/m = \frac{10}{2} = 5$ so $\lambda = \frac{5}{2}$

$$\omega^2 = \frac{k}{m} = \frac{12}{m} = 6$$

$$\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$$

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

The ODE is $mx'' + \beta x' + kx = 0$

Here, $m = 3$, $k = 12$ and $\beta = 12$ so

$$3x'' + 12x' + 12x = 0$$

$$x'' + 4x' + 4x = 0$$

The characteristic eqn is $r^2 + 4r + 4 = 0$

$(r+2)^2=0 \implies r=-2$ repeated

The system is critically damped.

Our IVP is

at equilibrium

upward 1m/sec

$$x''+4x'+4x=0 \qquad x(0)=0 \qquad x'(0)=1$$

The general solution is

$$x(t)=c_1e^{-2t}+c_2te^{-2t}$$

$$x'(t)=-2c_1e^{-2t}+c_2e^{-2t}-2c_2te^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 0 \implies c_1 = 0$$

$$x'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 e^0 = 1$$

$$c_2 = 1$$

The displacement for all $t > 0$ is

$$x(t) = t e^{-2t}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$x'' + \omega^2 x = F_0 \sin(\gamma t)$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma \neq \omega$, this would be the right form. The final solution would look like

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t)$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A\cos(\gamma t) + B\sin(\gamma t)$$

If $\gamma = \omega$, this is not the correct form. The correct form for x_p would be

$x_p = At\cos(\omega t) + Bt\sin(\omega t)$ * ω is γ *

The amplitude grows with time.

Forced Undamped Motion and Resonance

$\gamma \neq \omega$

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.