October 17 Math 2306 sec. 56 Fall 2017

Section 11: Linear Mechanical Equations

Simple Harmonic Motion: In the absence of any damping or external driving force, we determined the displacement x from equilibrium of an object suspended from a spring according to Hooke's law:

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The value

$$\omega^2 = \frac{k}{m}$$

where k is the spring constant and m the mass of the suspended object.

The equation of motion

The solution to the IVP $x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$ is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.**

We took the sign convention that the direction up is positive ($x > 0$) and down is negative ($x < 0$).

A Couple of Terms:

From Equilibrium: If we're told that motion starts **from equilibirum**, this means the initial position is equilibrium. That is

$$x(0) = 0 \quad \text{i.e.} \quad x_0 = 0.$$

From Rest: If we're told that motion starts **from rest**, this means that the initial velocity is zero. That is,

$$x'(0) = 0 \quad \text{i.e.} \quad x_1 = 0.$$

Free Damped Motion

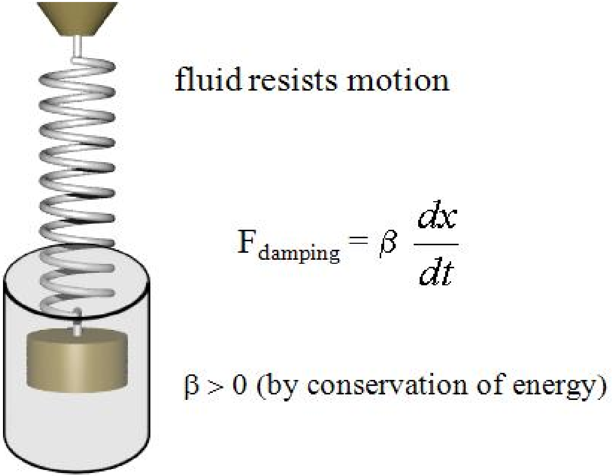

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

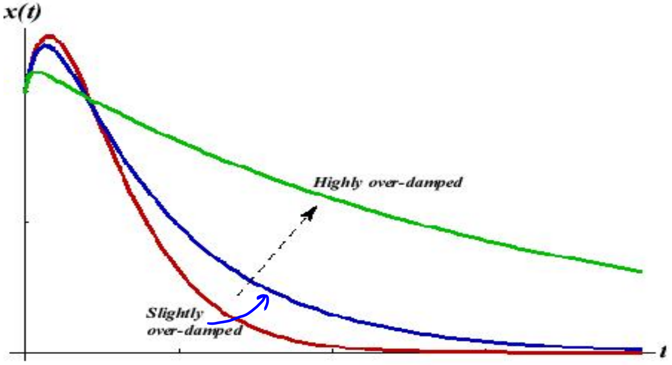

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

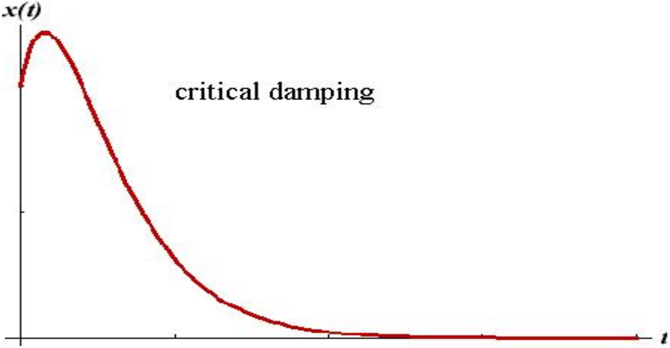

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

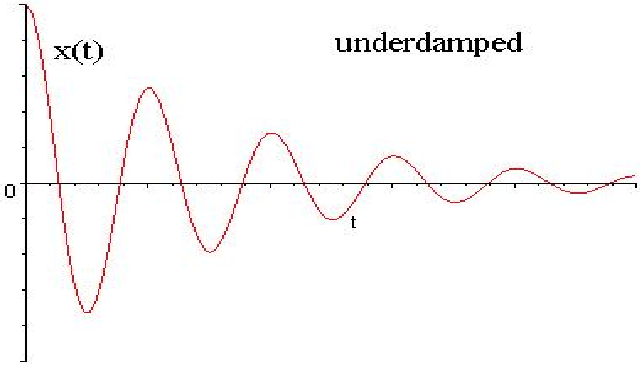

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

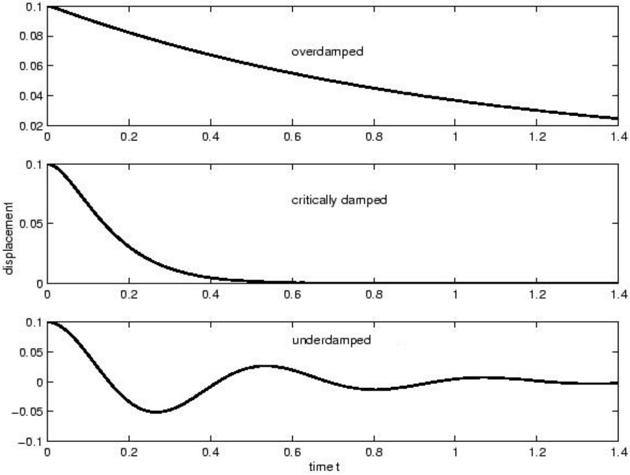

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The equation is $mx'' + \beta x' + kx = 0$

m (mass) $m = 2$, k (spring constant) $k = 12$

β (damping coefficient) $\beta = 10$

The ODE is $2x'' + 10x' + 12x = 0$

Standard form: $x'' + 5x' + 6x = 0$

Char. Eqn. $r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0$

$r_1 = -2, \quad r_2 = -3$

2 distinct real roots, the system is

over damped

Note here that $2\lambda = 5$ and $\omega^2 = 6$.

so $\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$.

Again we conclude that the system is over damped.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

Here $m = 3$, $\beta = 12$, $k = 12$

The eqn is $3x'' + 12x' + 12x = 0$

Standard form: $x'' + 4x' + 4x = 0$

Char. Eqn $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$

$r = -2$ repeated

This system is critically damped.

The initial conditions are

$$x(0)=0 \qquad x'(0)=1$$ (upward, so positive)

From the characteristic eqn.

$$x_1(t)=e^{-2t}, \quad x_2(t)=te^{-2t}$$

with general solution

$$x=c_1e^{-2t}+c_2te^{-2t}$$

$$x'(t)=-2c_1e^{-2t}+c_2e^{-2t}-2c_2te^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$x'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 e^0 = 1$$

$$c_2 = 1$$

The equation of motion (i.e. solution to the IVP) is $x(t) = te^{-2t}$.

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Note that this is a linear, constant coefficient, nonhomogeneous equation. Find x_c from the characteristic equation, then find x_p using either the method of Undetermined Coefficients or Variation of Parameters.

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and } (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$

If $\gamma \neq \omega$, Then this form does not match x_c. This form will work. The solution will be a sum of sines / cosines of ωt and sines / cosines of γt.

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma = \omega$, then this matches x_c. The guess must be modified so the correct form of x_p is $x_p = At\cos(\omega t) + Bt\sin(\omega t)$.

The amplitude of such a function grows without bound. This is resonance.

Natural frequency ω = resonance frequency.

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

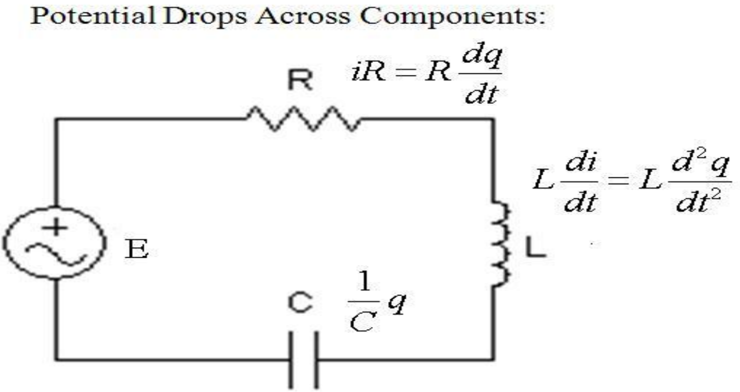

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{lrcl} \textbf{overdamped if} & R^2 - 4L/C & > & 0, \\ \textbf{critically damped if} & R^2 - 4L/C & = & 0, \\ \textbf{underdamped if} & R^2 - 4L/C & < & 0. \end{array}$$

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

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From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The equation is $Lq'' + Rq' + \frac{1}{C} q = E$

$L = \frac{1}{2}$, $R = 10$, $C = 4 \cdot 10^{-3}$

$$\frac{1}{2} q'' + 10 q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t)$$

$$2 \cdot \frac{1}{4 \cdot 10^{-3}} = \frac{2 \cdot 10^3}{4} = \frac{10^3}{2} = \frac{1000}{2} = 500$$

In standard form:

$$q'' + 20q' + 500q = 10\cos(10t)$$

Find q_c: Char. Eqn $r^2 + 20r + 500 = 0$

$$r^2 + 20r + 100 + 400 = 0$$

$$(r+10)^2 = -400$$

$$r + 10 = \pm 20i$$

$$r = -10 \pm 20i \qquad \alpha = -10,\ \beta = 20$$

$$q_1 = e^{-10t}\cos(20t), \quad q_2 = e^{-10t}\sin(20t)$$

$$q_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$$

Using undetermined coefficients to find q_p

Guess $q_p = A\cos(10t) + B\sin(10t)$ — This will work.

$$q_p' = -10A\sin(10t) + 10B\cos(10t)$$

$$q_p'' = -100A\cos(10t) - 100B\sin(10t)$$

$$q_p'' + 20q_p' + 500q_p =$$

$$-100A\cos(10t) - 100B\sin(10t) - 200A\sin(10t) + 200B\cos(10t) +$$

$$500A\cos(10t) + 500B\sin(10t) = 10\cos(10t)$$

$$\cos(10t)\left(-100A+200B+500A\right)+\sin(10t)\left(-100B-200A+500B\right)$$

$$=10\cos(10t)+0\cdot\sin(10t)$$

$$400A+200B=10$$

$$-200A+400B=0 \Rightarrow 200A=400B \Rightarrow A=2B$$

$$400(2B)+200B=10 \Rightarrow 1000B=10$$

$$B=\frac{10}{1000}=\frac{1}{100}=0.01$$

$$A=2B=\frac{2}{100}=0.02$$

The steady state Charge

$$q_p = 0.02 \cos(10t) + 0.01 \sin(10t).$$

The steady state current i_p is $\frac{dq_p}{dt}$

$$i_p = -0.2 \sin(10t) + 0.1 \cos(10t).$$