October 17 Math 2306 sec. 57 Fall 2017

Section 11: Linear Mechanical Equations

Simple Harmonic Motion: In the absence of any damping or external driving force, we determined the displacement x from equilibrium of an object suspended from a spring according to Hooke's law:

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The value

$$\omega^2 = \frac{k}{m}$$

where k is the spring constant and m the mass of the suspended object.

The equation of motion

The solution to the IVP $x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$ is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.**

We took the sign convention that the direction up is positive ($x > 0$) and down is negative ($x < 0$).

Terms: From Equilibrium : means $x(0) = 0$

From Rest : means $x'(0) = 0$

Free Damped Motion

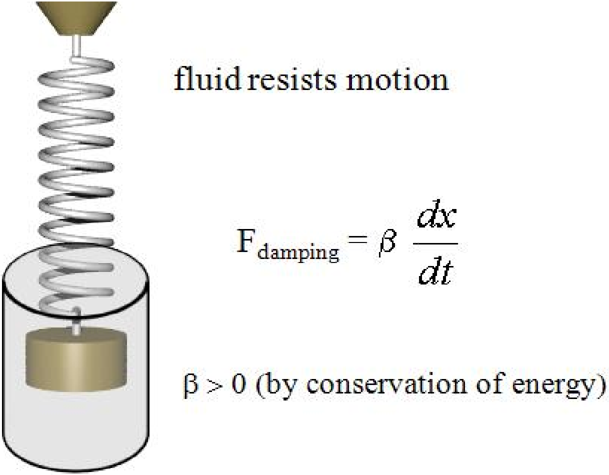

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

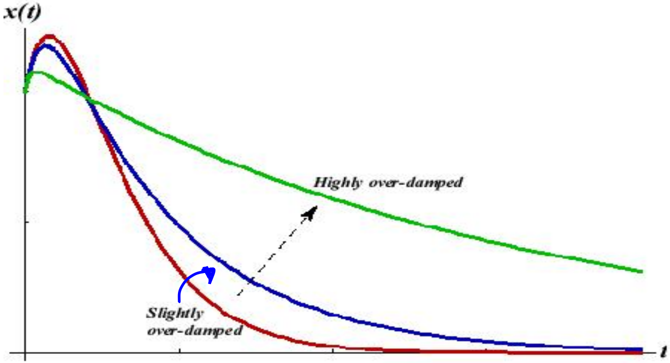

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

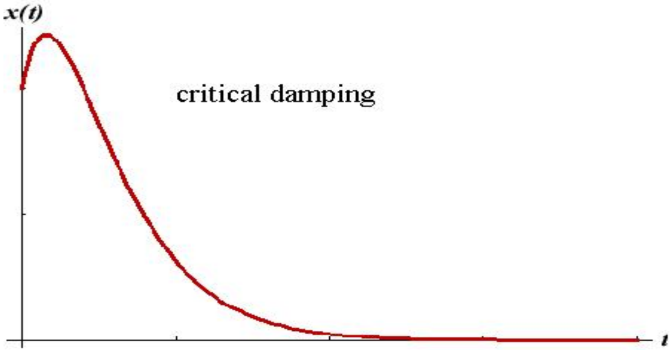

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

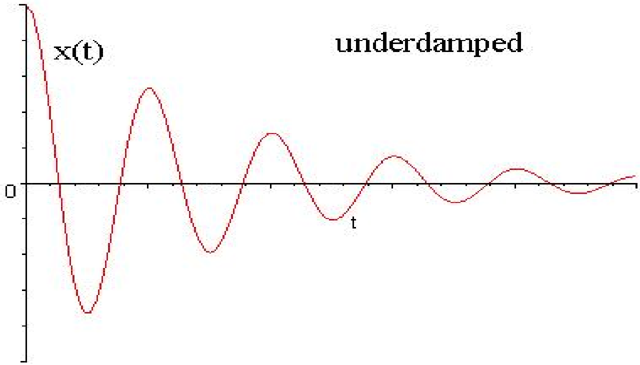

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

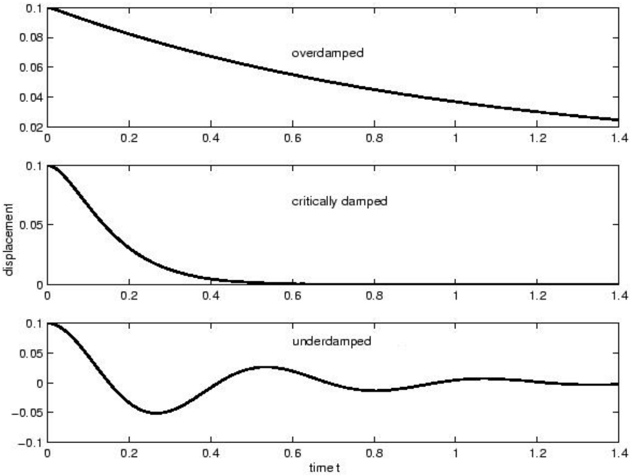

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The form of the ODE is $mx'' + \beta x' + kx = 0$

mass $m = 2$, spring constant $k = 12$

damping coefficient $\beta = 10$. The ODE is

$$2x'' + 10x' + 12x = 0$$

Standard form

$$x'' + 5x' + 6x = 0$$

Charac. Eqn $r^2+5r+6=0$

$(r+2)(r+3)=0 \Rightarrow r=-2$ or $r=-3$.

2 distinct real roots.
The system is overdamped.

Here $2\lambda=5$ and $\omega^2=6$

$$\lambda^2-\omega^2=\left(\frac{5}{2}\right)^2-6=\frac{25}{4}-\frac{24}{4}=\frac{1}{4}>0$$

i.e. $\lambda^2>\omega^2$ so again, the system is overdamped.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$mx'' + \beta x' + kx = 0 \qquad m = 3, \quad k = 12, \quad \beta = 12$

$3x'' + 12x' + 12x = 0$

Standard form $\quad x'' + 4x' + 4x = 0$

Char. Eqn $\quad r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$

$r = -2$ repeated.

One real root : The system is critically damped.

From the characteristic equation

$$x_1 = e^{-2t}, \quad x_2 = te^{-2t}$$

with general solution

$$x = c_1 e^{-2t} + c_2 te^{-2t}$$

We're given $x(0) = 0$ $\quad x'(0) = 1$ (upward so positive)

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 te^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 0 \implies c_1 = 0$$

$$x'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 e^0 = 1$$

$$c_2 = 1$$

The equation of motion (solution to the IVP) is

$$\boxed{x = te^{-2t}.}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

2nd order, linear, constant coefficient, nonhomogeneous ODE.
Get x_c from the characteristic Eqn. Get x_p from Undetermined coefficients or Variation of parameters.

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and } (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma \neq \omega$, this form works.

The general solution will be a sum of sines/cosines of ωt and sines/cosines of γt.

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma = \omega$, this guess would have to be modified as it matches x_c. The correct form of x_p would be

$$x_p = At\cos(\gamma t) + Bt\sin(\gamma t).$$

The amplitude can grow without bound.

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

natural frequency ω = resonance frequency

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega}t\cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

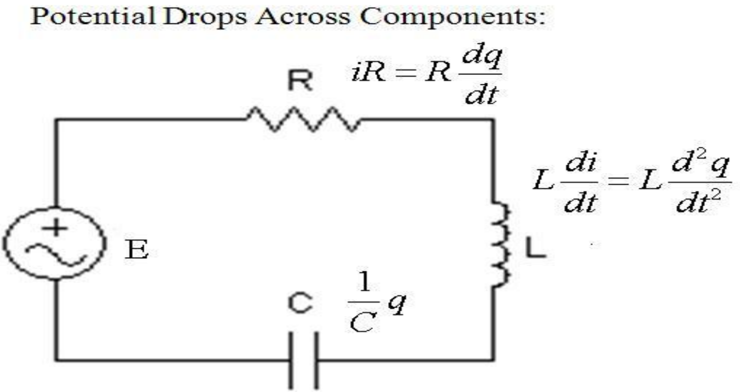

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

2nd order, linear, constant coefficient, nonhomogeneous (if $E \neq 0$) ODE.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{llcl} \textbf{overdamped} \text{ if} & R^2 - 4L/C & > & 0, \\ \textbf{critically damped} \text{ if} & R^2 - 4L/C & = & 0, \\ \textbf{underdamped} \text{ if} & R^2 - 4L/C & < & 0. \end{array}$$

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

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$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

$$Lq'' + Rq' + \frac{1}{C} q = E \qquad L = \frac{1}{2}, \quad R = 10, \quad C = 4 \cdot 10^{-3}$$

$$E = 5\cos(10t) \qquad \frac{1}{2} q'' + 10 q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t)$$

Note $\quad 2 \cdot \frac{1}{4 \cdot 10^{-3}} = \frac{2 \cdot 10^3}{4} = \frac{10^3}{2} = \frac{1000}{2} = 500$

Standard form: $\quad q'' + 20q' + 500q = 10\cos(10t)$

Charac. eqn $\quad r^2 + 20r + 500 = 0$

$$r^2 + 20r + 100 + 400 = 0$$

$$(r+10)^2 = -400 \implies r + 10 = \pm 20i$$

$$r = -10 \pm 20i \qquad \alpha = -10, \ \beta = 20$$

$$y_1 = e^{-10t}\cos(20t), \quad y_2 = e^{-10t}\sin(20t)$$

$$y_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$$

Using undetermined coefficients, guess

$$y_p = A\cos(10t) + B\sin(10t)$$ will work

$$y_p' = -10A\sin(10t) + 10B\cos(10t)$$

$$y_p'' = -100A\cos(10t) - 100B\sin(10t)$$

$$y_p'' + 20y_p' + 500y_p =$$

$$-100A\cos(10t) - 100B\sin(10t) - 200A\sin(10t) + 200B\cos(10t)$$

$$+ 500A\cos(10t) + 500B\sin(10t) = 10\cos(10t)$$

$$\cos(10t)\left(\underline{-100A + 200B + 500A}\right) + \sin(10t)\left(\underline{-100B - 200A + 500B}\right)$$

$$= \underline{10}\cos(10t) + \underline{0}\cdot\sin(10t)$$

$$400A + 200B = 10$$

$$-200A + 400B = 0 \implies A = 2B$$

$$400(2B) + 200B = 10$$

$$1000B = 10 \implies B = \frac{10}{1000} = 0.01$$

So $A = 2B = 0.02$.

The steady state charge is

$$q_p = 0.02\cos(10t) + 0.01\sin(10t)$$

The steady state current $i_p = \frac{dq_p}{dt}$

is

$$i_p = -0.2\sin(10t) + 0.1\cos(10t).$$