October 12 Math 3260 sec. 57 Fall 2017

Section 6.1: Inner Product, Length, and Orthogonality

Recall that we defined the inner product in $\mathbb{R}^n$: **Definition:** For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ we define the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

We noted that this product has several properties.

Theorem (Properties of the Inner Product)

We'll use the notation $\quad \mathbf{u}\cdot\mathbf{v} \;=\; \mathbf{u}^T\mathbf{v}.$

Theorem: For $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and real scalar c

(a) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$

(c) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$

(d) $\mathbf{u}\cdot\mathbf{u}\geq 0$, with $\mathbf{u}\cdot\mathbf{u}=0$ if and only if $\mathbf{u}=\mathbf{0}$.

The Norm and Orthogonality

Definition: The **norm** of the vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative number denoted and defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$.

If $\mathbf{v}$ is any nonzero vector, the vector $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector in the direction of $\mathbf{v}$.

Definition: Two vectors are $\mathbf{u}$ and $\mathbf{v}$ **orthogonal** if $\mathbf{u}\cdot\mathbf{v} = 0$.

In $\mathbb{R}^n$, orthogonality corresponds geometrically with begin perpendicular.

Orthogonal Complement

Definition: Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{z}$ in $\mathbb{R}^n$ is said to be **orthogonal to** W if $\mathbf{z}$ is orthogonal to every vector in W.

$\vec{z}$ is orthogonal to W if
$\vec{z} \cdot \vec{w} = 0$ for every $\vec{w}$ in W.

Definition: Given a subspace W of $\mathbb{R}^n$, the set of all vectors orthogonal to W is called the **orthogonal complement** of W and is denoted by

$$W^{\perp}.$$

Theorem:

$W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Note that for any $\vec{w}$ in W.

$$\vec{w}\cdot\vec{0} = w_1\cdot 0 + w_2\cdot 0 + \cdots + w_n\cdot 0 = 0$$

so $\vec{0}$ is orthogonal to W; it is in $W^{\perp}$.

Suppose $\vec{u}$ and $\vec{v}$ are in $W^{\perp}$. Then $\vec{u}\cdot\vec{w}=0$ and $\vec{v}\cdot\vec{w}=0$ for every $\vec{w}$ in W. Note

$$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w} = 0+0 = 0$$

So $\vec{u}+\vec{v}$ is in $W^{\perp}$. $W^{\perp}$ is closed under vector addition.

For scalar c and $\vec{w}$ in W

$$(c\vec{u})\cdot\vec{w} = c(\vec{u}\cdot\vec{w}) = c(0) = 0$$

Hence $W^\perp$ is closed under scalar multiplication.

$W^\perp$ meets all properties required to be a subspace of $\mathbb{R}^n$.

Example

Let W =Span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Show that $W^{\perp}$ =Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Give a geometric interpretation of W and $W^{\perp}$ as subspaces of $\mathbb{R}^3$.

If $\vec{w}$ is in W, then $\vec{w} = w_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} w_1 \\ 0 \\ w_3 \end{bmatrix}$

for some real numbers w_1 and w_3. For $\vec{z}$ in $\mathbb{R}^3$,

$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$, if $\vec{z} \cdot \vec{w} = 0$, then $\vec{z}$ is in $W^{\perp}$.

$0 = \vec{z} \cdot \vec{w} = z_1 w_1 + 0 + z_3 w_3$. This has to hold

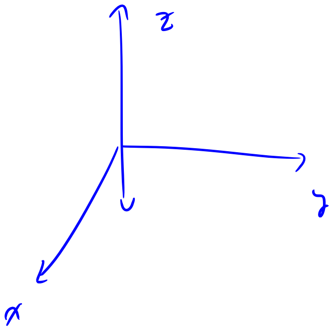
for every w_1, w_3 pair. If $w_1 = 1$ and $w_3 = 0$, we get $z_1 \cdot 1 = 0 \Rightarrow z_1 = 0$. Taking $w_1 = 0, w_3 = 1$, we get $0 = z_3 \cdot 1 \Rightarrow z_3 = 0$. So $\vec{z}$ must have the form $\vec{z} = \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix} = z_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence

$$W^\perp = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

As for the geometry, $\vec{w}$ in W looks like $(w_1, 0, w_3)$. These are points in the xz plane.

Points in $W^\perp$ look like $(0, z_2, 0)$.

This is the y-axis.



Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$. Show that if $\mathbf{x}$ is in $\mathrm{Nul}(A)$, then $\mathbf{x}$ is in $[\mathrm{Row}(A)]^{\perp}$.

If $\vec{x}$ is in Nul A, then $A\vec{x} = \vec{0}$. For such $\vec{x}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 + 2x_3 = \vec{x} \cdot \vec{v}_1$ where $\vec{v}_1$ is the first row of A

and $-2x_1 + 4x_3 = \vec{x} \cdot \vec{v}_2$ where $\vec{v}_2$ is the 2^{nd} row of A.

Since $\text{row}\, A = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, any $\vec{x}$ in $\text{Nul}\, A$ is orthogonal to every vector in $\text{row}\, A$.

Theorem

Theorem: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A. That is

$$[\mathrm{Row}(A)]^\perp \;=\; \mathrm{Nul}(A).$$

The orthongal complement of the column space of A is the null space of A^T—i.e.

$$[\mathrm{Col}(A)]^\perp \;=\; \mathrm{Nul}(A^T).$$

Example: Find the orthogonal complement of Col(A)

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -3 & 3 & 0 \\ 2 & 4 & 1 \\ 2 & -2 & 9 \\ 0 & 1 & -1 \end{bmatrix}$$

$$[\text{Col } A]^{\perp} = \text{Nul}(A^T)$$

$$A^T = \begin{bmatrix} 5 & -3 & 2 & 2 & 0 \\ 2 & 3 & 4 & -2 & 1 \\ 1 & 0 & 1 & 9 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -54 & 7 \\ 0 & 1 & 0 & -\frac{146}{3} & 19/3 \\ 0 & 0 & 1 & 63 & -8 \end{bmatrix}$$

For $\vec{x}$ in Nul A^T

$$x_1 = 54x_4 - 7x_5$$

$$x_2 = \frac{146}{3}x_4 - \frac{19}{3}x_5$$

$$x_3 = -63x_4 + 8x_5$$

x_4, x_5 - free

$$\vec{x} = x_4 \begin{bmatrix} 54 \\ \frac{146}{3} \\ -63 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -\frac{19}{3} \\ 8 \\ 0 \\ 1 \end{bmatrix}$$

so

$$[\text{Col } A]^{\perp} = \text{Span} \left\{ \begin{bmatrix} 54 \\ \frac{146}{3} \\ -63 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -\frac{19}{3} \\ 8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Section 6.2: Orthogonal Sets

Remark: We know that if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace W of $\mathbb{R}^n$, then each vector $\mathbf{x}$ in W can be realized (uniquely) as a sum

$$\mathbf{x} = c_1\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p.$$

If n is very large, the computations needed to determine the coefficients $c_1, \ldots, c_p$ may require a lot of time (and machine memory).

Question: Can we seek a basis whose nature simplifies this task? And what properties should such a basis possess?

Orthogonal Sets

Definition: An indexed set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** provided each pair of distinct vectors in the set is orthogonal. That is, provided

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{whenever} \quad i \neq j.$$

Example: Show that the set $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$ is an orthogonal set.

label those $\vec{u}_1$ $\vec{u}_2$ $\vec{u}_3$

$\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1(2) + 1(1) = -3 + 2 + 1 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = 3(-1) + 1(-4) + 1(7) = -3 - 4 + 7 = 0$

$$\vec{u}_2 \cdot \vec{u}_3 = -1(-1) + 2(-4) + 1\cdot(7) = 1 - 8 + 7 = 0$$

So $\vec{u}_i \cdot \vec{u}_j = 0$ for all $i \neq j$. The set is orthogonal.

Orthongal Basis

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis that is also an orthogonal set.

Theorem: Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then each vector $\mathbf{y}$ in W can be written as the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p, \quad \text{where the weights}$$

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j}.$$

Take u_j

$$\vec{u}_j \cdot \vec{y} = \vec{u}_j \cdot \left(c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_j \vec{u}_j + \dots + c_p \vec{u}_p \right)$$

$$= c_1 \vec{u}_1 \cdot \vec{u}_j + c_2 \vec{u}_2 \cdot \vec{u}_j + \dots + c_j \vec{u}_j \cdot \vec{u}_j + \dots + c_p \vec{u}_p \cdot \vec{u}_j$$

$$= 0 + 0 + \dots + c_j \|\vec{u}_j\|^2 + 0 \dots$$

$$c_j = \frac{\vec{u}_j \cdot \vec{y}}{\|\vec{u}_j\|^2}$$

Example

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$$ is an orthogonal basis of $\mathbb{R}^3$. Express

$\vec{u}_1$ $\quad$ $\vec{u}_2$ $\quad$ $\vec{u}_3$

the vector $\mathbf{y} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of the basis vectors.

$\|\vec{u}_1\|^2 = 9+1+1 = 11, \quad \|\vec{u}_2\|^2 = 1+4+1 = 6, \quad \|\vec{u}_3\|^2 = 1+16+49 = 66$

$\vec{y}\cdot\vec{u}_1 = -6+3 = -3, \quad \vec{y}\cdot\vec{u}_2 = 2+6 = 8, \quad \vec{y}\cdot\vec{u}_3 = 2-12 = -10$

$$\vec{y} = \frac{-3}{11}\vec{u}_1 + \frac{8}{6}\vec{u}_2 - \frac{10}{66}\vec{u}_3$$

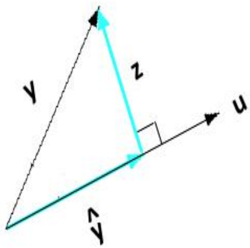
$$= \frac{-3}{11}\vec{u}_1 + \frac{4}{3}\vec{u}_2 - \frac{5}{33}\vec{u}_3 = \frac{-9\vec{u}_1 + 44\vec{u}_2 - 5\vec{u}_3}{33}$$

Projection

Given a nonzero vector **u**, suppose we wish to decompose another nonzero vector **y** into a sum of the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

in such a way that $\hat{\mathbf{y}}$ is parallel to **u** and **z** is perpendicular to **u**.



Projection

Since $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$, there is a scalar α such that

$$\hat{\mathbf{y}} = \alpha\mathbf{u}.$$

From $\vec{y} = \hat{y} + \vec{z}$, $\hat{y} = \vec{y} - \vec{z}$. Dot product w/ $\vec{u}$

$$\vec{u}\cdot\hat{y} = \vec{u}\cdot(\vec{y} - \vec{z}) = \vec{u}\cdot\vec{y} - \vec{u}\cdot\vec{z}$$

$$\vec{u}\cdot(\alpha\vec{u}) = \vec{u}\cdot\vec{y} - 0$$

$$\alpha(\vec{u}\cdot\vec{u}) = \vec{u}\cdot\vec{y} \Rightarrow \alpha = \frac{\vec{u}\cdot\vec{y}}{\|\vec{u}\|^2}$$