Section 6.1: Inner Product, Length, and Orthogonality

Recall that we defined the inner product in $\mathbb{R}^n$: **Definition:** For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ we define the inner product of $\mathbf{u}$ and $\mathbf{v}$ (also called the dot product) by the matrix product

$$\mathbf{u}^T \mathbf{v} = [u_1 \ u_2 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

We noted that this product has several properties.
Theorem (Properties of the Inner Product)

We’ll use the notation \( \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} \).

**Theorem:** For \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) in \( \mathbb{R}^n \) and real scalar \( c \)

(a) \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)

(b) \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \)

(c) \( c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) \)

(d) \( \mathbf{u} \cdot \mathbf{u} \geq 0 \), with \( \mathbf{u} \cdot \mathbf{u} = 0 \) if and only if \( \mathbf{u} = \mathbf{0} \).
The Norm and Orthogonality

**Definition:** The **norm** of the vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative number denoted and defined by

$$
\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}
$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$.

If $\mathbf{v}$ is any nonzero vector, the vector $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector in the direction of $\mathbf{v}$.

**Definition:** Two vectors are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. In $\mathbb{R}^n$, orthogonality corresponds geometrically with begin perpendicular.
Orthogonal Complement

Definition: Let $W$ be a subspace of $\mathbb{R}^n$. A vector $z$ in $\mathbb{R}^n$ is said to be orthogonal to $W$ if $z$ is orthogonal to every vector in $W$.

\[ \bar{z} \cdot \bar{w} = 0 \text{ for every } \bar{w} \text{ in } W. \]

Definition: Given a subspace $W$ of $\mathbb{R}^n$, the set of all vectors orthogonal to $W$ is called the \textbf{orthogonal complement} of $W$ and is denoted by

\[ W^\perp. \]
Theorem:

$W^\perp$ is a subspace of $\mathbb{R}^n$.

Note, for any $\bar{w}$ in $W$, $\bar{o} \cdot \bar{w} = 0 \cdot w_1 + 0 \cdot w_2 + \ldots + 0 \cdot w_n = 0$.

So $\bar{o}$ is in $W^\perp$.

If $\bar{u}$ and $\bar{v}$ are in $W^\perp$, then

$\bar{u} \cdot \bar{w} = 0$ and $\bar{v} \cdot \bar{w} = 0$ for every $\bar{w}$ in $W$.

Then $(\bar{u} + \bar{v}) \cdot \bar{w} = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w} = 0 + 0 = 0$.

$W^\perp$ is closed under vector addition.

For scalar $c$ and $\bar{w}$ in $W$

$(c\bar{u}) \cdot \bar{w} = c(\bar{u} \cdot \bar{w}) = c(0) = 0$
$W^\perp$ is closed under scalar multiplication too. Hence $W^\perp$ is a subspace of $\mathbb{R}^n$. 
Example

Let \( W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \). Show that \( W^\perp = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \).

Give a geometric interpretation of \( W \) and \( W^\perp \) as subspaces of \( \mathbb{R}^3 \).

Any vector \( \tilde{w} \) in \( W \) has the form

\[
\tilde{w} = w_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} w_1 \\ 0 \\ w_2 \end{bmatrix}.
\]

If \( \tilde{z} \) is in \( W^\perp \), then

\[
\tilde{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},
\]

so

\[
\tilde{z} \cdot \tilde{w} = 0 = z_1 w_1 + z_3 w_3.
\]

This has to hold for all \( w_1, w_3 \) pairs. If \( w_1 = 1 \) and \( w_3 = 0 \), we get

\[
0 = z_1 \cdot 1 + z_3 \cdot 0 \Rightarrow z_1 = 0.
\]
Taking $w_1 = 0$ and $w_3 = 1$, we'd get

$$0 = z_3 \cdot 1 \implies z_3 = 0.$$  

The form of $\tilde{z}$ must be $\tilde{z} = z_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, hence

$$w^\perp = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$  

Geometrically, $W$ is the set of vectors $(x, 0, 2)$. These points constitute the $xz$-plane. The vectors in $W^\perp$ look like $(0, y, 0)$. 
This is the y-axis.
Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$. Show that if $x$ is in $\text{Nul}(A)$, then $x$ is in $[\text{Row}(A)]^\perp$.

$$ \text{Row } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \right\}. $$

If $x$ is in $\text{Nul}(A)$, then $Ax = 0$. For $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$$ A \hat{x} = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. $$

$$ \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. $$
The entries are the inner products of $\tilde{x}$ with the rows of $A$. So $\tilde{x}$ is orthogonal to each row in $A$. 
Theorem: Let $A$ be an $m \times n$ matrix. The orthogonal complement of the row space of $A$ is the null space of $A$. That is

$$[\text{Row}(A)]^\perp = \text{Nul}(A).$$

The orthogonal complement of the column space of $A$ is the null space of $A^T$—i.e.

$$[\text{Col}(A)]^\perp = \text{Nul}(A^T).$$
Example: Find the orthogonal complement of $\text{Col}(A)$

$A = \begin{bmatrix}
5 & 2 & 1 \\
-3 & 3 & 0 \\
2 & 4 & 1 \\
2 & -2 & 9 \\
0 & 1 & -1 \\
\end{bmatrix}$

$[\text{col } A]^{\perp} = \text{null}(A^\top)$

$[\begin{bmatrix} 5 & -3 & 2 & 2 & 0 \\
2 & 3 & 4 & -2 & 1 \\
1 & 0 & 1 & 9 & -1 \\
\end{bmatrix}]$

$\vec{x} = \begin{bmatrix} 1 & 0 & 0 & -54 & 7 \\
0 & 1 & 0 & -146 & 19 \\
0 & 0 & 1 & 63 & -8 \\
\end{bmatrix}$

If $A^\top \vec{x} = 0$, then

$x_1 = 54 \times x_4 - 7 \times x_5$

$x_2 = \frac{146}{3} \times x_4 - \frac{19}{3} \times x_5$
\[ x_3 = -63x_4 + 8x_5 \]

\( x_4, x_5 \) free

\[
\begin{align*}
\hat{x} &= x_4 \begin{bmatrix} 54 \\ 146/3 \\ -63 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -19/3 \\ 8 \\ 0 \\ 1 \end{bmatrix} \\
\end{align*}
\]

\[
[\text{Col}(A)]^T = \text{Span} \left\{ \begin{bmatrix} 54 \\ 146/3 \\ -63 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -19/3 \\ 8 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]