October 12 Math 3260 sec. 58 Fall 2017

Section 6.1: Inner Product, Length, and Orthogonality

Recall that we defined the inner product in $\mathbb{R}^n$: **Definition:** For vectors **u** and **v** in $\mathbb{R}^n$ we define the **inner product** of **u** and **v** (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

We noted that this product has several properties.

Theorem (Properties of the Inner Product)

We'll use the notation $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v}$.

Theorem: For $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and real scalar c

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(c) $c(\mathbf{u}\cdot\mathbf{v}) = (c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v})$

(d) $\mathbf{u}\cdot\mathbf{u} \geq 0$, with $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

The Norm and Orthogonality

Definition: The **norm** of the vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative number denoted and defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$.

If $\mathbf{v}$ is any nonzero vector, the vector $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector in the direction of $\mathbf{v}$.

Definition: Two vectors are $\mathbf{u}$ and $\mathbf{v}$ **orthogonal** if $\mathbf{u}\cdot\mathbf{v} = 0$.

In $\mathbb{R}^n$, orthogonality corresponds geometrically with begin perpendicular.

Orthogonal Complement

Definition: Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{z}$ in $\mathbb{R}^n$ is said to be **orthogonal to** W if $\mathbf{z}$ is orthogonal to every vector in W.

$\vec{z} \cdot \vec{w} = 0$ for every $\vec{w}$ in W.

Definition: Given a subspace W of $\mathbb{R}^n$, the set of all vectors orthogonal to W is called the **orthogonal complement** of W and is denoted by

$$W^\perp.$$

Theorem:

$W^\perp$ is a subspace of $\mathbb{R}^n$.

Note, for any $\vec{w}$ in W, $\vec{0}\cdot\vec{w} = 0w_1 + 0w_2 + \cdots + 0w_n = 0$.

So $\vec{0}$ is in $W^\perp$.

If $\vec{u}$ and $\vec{v}$ are in $W^\perp$, then

$$\vec{u}\cdot\vec{w} = 0 \quad \text{and} \quad \vec{v}\cdot\vec{w} = 0 \quad \text{for every } \vec{w} \text{ in } W.$$

Then $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w} = 0 + 0 = 0$.

$W^\perp$ is closed under vector addition.

For scalar c and $\vec{w}$ in W

$$(c\vec{u})\cdot\vec{w} = c(\vec{u}\cdot\vec{w}) = c(0) = 0$$

$W^\perp$ is closed under scalar multiplication too. Hence $W^\perp$ is a subspace of $\mathbb{R}^n$.

Example

Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Show that $W^\perp = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Give a geometric interpretation of W and $W^\perp$ as subspaces of $\mathbb{R}^3$.

Any vector $\vec{w}$ in W has the form

$$\vec{w} = w_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} w_1 \\ 0 \\ w_2 \end{bmatrix}.$$

If $\vec{z}$ is in $W^\perp$ where $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$, then

$$\vec{z} \cdot \vec{w} = 0 = z_1 w_1 + z_3 w_3.$$

This has to hold for all w_1, w_3 pairs. If $w_1 = 1$ and $w_3 = 0$, we get

$$0 = z_1 \cdot 1 + z_3 \cdot 0 \implies z_1 = 0.$$

Taking $w_1 = 0$ and $w_3 = 1$, we'd get

$$0 = z_3 \cdot 1 \implies z_3 = 0.$$

The form of $\vec{z}$ must be $\vec{z} = z_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, hence

$$W^{\perp} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Geometrically, W is the set of vectors $(x, 0, z)$. These points constitute the xz-plane.

The vectors in $W^{\perp}$ look like $(0, y, 0)$.

This is the y-axis.

Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$. Show that if **x** is in Nul(A), then **x** is in $[\text{Row}(A)]^{\perp}$.

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

If $\vec{x}$ is in Nul(A), then $A\vec{x} = \vec{0}$. For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The entries are the inner products of $\vec{x}$ with the rows of A. So $\vec{x}$ is orthogonal to each row in A.

Theorem

Theorem: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A. That is

$$[\mathrm{Row}(A)]^\perp \ = \ \mathrm{Nul}(A).$$

The orthongal complement of the column space of A is the null space of A^T—i.e.

$$[\mathrm{Col}(A)]^\perp \ = \ \mathrm{Nul}(A^T).$$

Example: Find the orthogonal complement of Col(A)

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -3 & 3 & 0 \\ 2 & 4 & 1 \\ 2 & -2 & 9 \\ 0 & 1 & -1 \end{bmatrix}$$

$[\mathrm{Col}\, A]^{\perp} = \mathrm{Nul}(A^T)$

$$\begin{bmatrix} 5 & -3 & 2 & 2 & 0 \\ 2 & 3 & 4 & -2 & 1 \\ 1 & 0 & 1 & 9 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -54 & 7 \\ 0 & 1 & 0 & -\frac{146}{3} & \frac{19}{3} \\ 0 & 0 & 1 & 63 & -8 \end{bmatrix}$$

If $A^T\vec{x} = \vec{0}$,

$x_1 = 54x_4 - 7x_5$

$x_2 = \frac{146}{3}x_4 - \frac{19}{3}x_5$

$$x_3 = -63x_4 + 8x_5$$

x_4, x_5 – free

$$\vec{x} = x_4 \begin{bmatrix} 54 \\ \frac{146}{3} \\ -63 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ \frac{-19}{3} \\ 8 \\ 0 \\ 1 \end{bmatrix}$$

$$[\mathrm{Col}(A)]^\perp = \mathrm{Span}\left\{ \begin{bmatrix} 54 \\ \frac{146}{3} \\ -63 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -19/3 \\ 8 \\ 0 \\ 1 \end{bmatrix} \right\}.$$