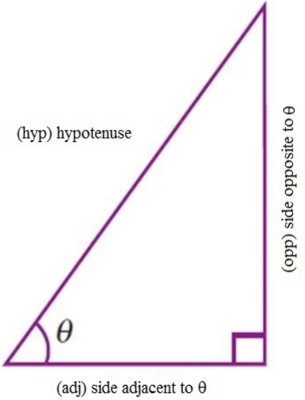
October 19 MATH 1113 sec. 51 Fall 2018

Sections 6.1 & 6.2: Trigonometric Functions of Acute Angles

We defined the six trigonometric values of an acute angle θ with reference to the triangle as labeled.



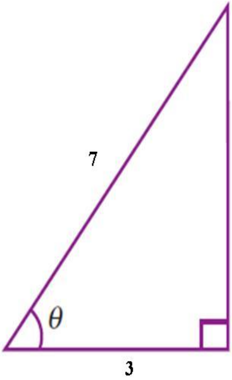
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

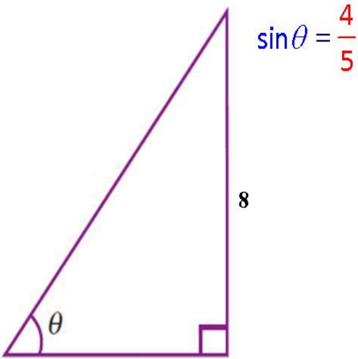
Example

Determine the six trigonometric values of the acute angle θ.



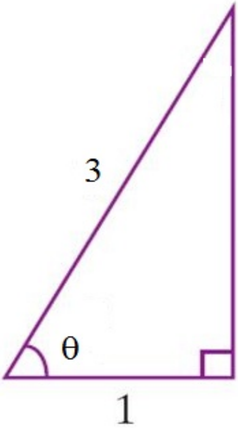
Example

Determine the six trigonometric values of the acute angle θ.



Question

For the angle θ shown, which statement is correct?



(a) $\sin\theta = \frac{\sqrt{8}}{3}$ and $\cos\theta = \frac{1}{3}$

(b) $\sin\theta = \frac{1}{3}$ and $\cos\theta = \frac{\sqrt{2}}{3}$

(c) $\tan\theta = \frac{1}{3}$ and $\sin\theta = \frac{\sqrt{2}}{3}$

(d) $\tan\theta = \sqrt{2}$ and $\cot\theta = \frac{1}{\sqrt{2}}$

Some Key Trigonometric Values

Use the triangles to determine the six trigonometric values of the angles $30^\circ, 45^\circ$, and 60°.

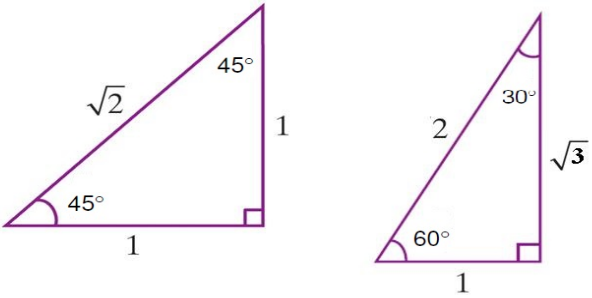


Figure: An isosceles right triangle of leg length 1 (left), and half of an equilateral triangle of side length 2 (right).

Commit To Memory

It is to our advantage to remember the following:

$$\sin 30^\circ = \frac{1}{2}, \qquad \sin 45^\circ = \frac{1}{\sqrt{2}}, \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}}, \qquad \cos 60^\circ = \frac{1}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}, \qquad \tan 45^\circ = 1, \qquad \tan 60^\circ = \sqrt{3}$$

We'll use these to find some other trigonometric values. Still others will require a calculator.

Calculator

Figure: Any scientific calculator will have built in functions for sine, cosine and tangent. (TI-84 shown)

Using a Calculator

Evaluate the following using a calculator. Round answers to three decimal places.

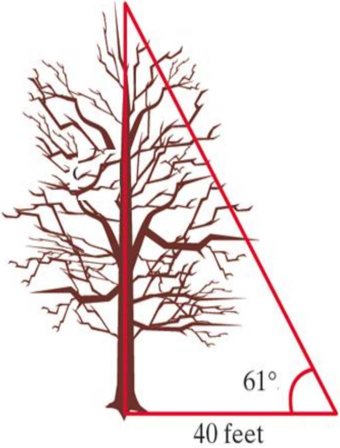
$\sin 16^\circ =$

$\sec 78.3^\circ =$

$\tan(65.4^\circ) =$

Application Example

Before cutting down a dead tree, you wish to determine its height. From a horizontal distance of 40 ft, you measure the angle of elevation from the ground to the top of the tree to be 61°. Determine the tree height to the nearest 100^{th} of a foot.



Application Example

The ramp of truck for moving touches the ground 14 feet from the end of the truck. If the ramp makes an angle of 28.5° with the ground, what is the length of the ramp?

Question

The ramp of truck for moving touches the ground 14 feet from the end of the truck. If the ramp makes an angle of 28.5° with the ground, what is the length of the ramp?

The length L of the ramp can be determined from the equation

(a) $\dfrac{L}{14} = \csc(28.5^\circ)$

(b) $\dfrac{L}{14} = \tan(28.5^\circ)$

(c) $\dfrac{14}{L} = \cos(28.5^\circ)$

(d) $\dfrac{14}{L} = \cot(28.5^\circ)$

Complementary Angles and Cofunction Identities

The two acute angles in a right triangle must sum to 90°. Two acute angles whose measures sum to 90° are called **complementary angles**. Given an acute angle θ its **complement** is the angle $90^\circ - \theta$.

Example Find the complementary angle of 27°.

Cofunction Identities

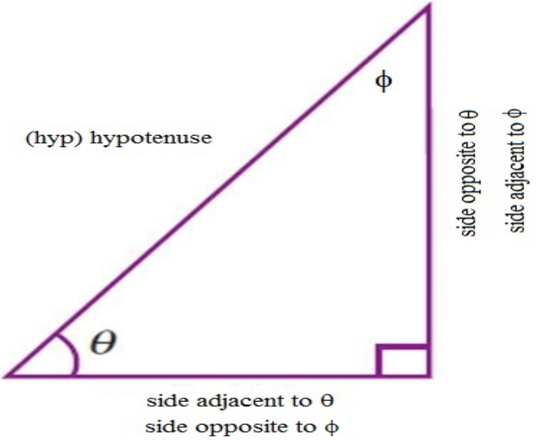


Figure: Note that for complementary angles θ and ϕ, the role of the legs (opposite versus adjacent) are interchanged.

Cofunction Identities

For any acute angle θ

$$\sin\theta = \cos(90^\circ - \theta) \qquad \cos\theta = \sin(90^\circ - \theta)$$

$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

$$\sec\theta = \csc(90^\circ - \theta) \qquad \csc\theta = \sec(90^\circ - \theta)$$

These equations define what are called **cofunction identities.**

Question

The value of $\sin(13^\circ)$ is equivalent to

(a) $\csc(77^\circ)$

(b) $\sin(77^\circ)$

(c) $\sec(77^\circ)$

(d) $\cos(77^\circ)$

Section 6.3: Angles, Rotations, and Angle Measures

We define an angle by a pair of rays (say R_1 and R_2) that share a common origin. We can indicate direction for an angle by indicating one ray as the **initial ray** and the other as the **terminal ray.**

We then define an angle as being **positive** if it is counter clock-wise and **negative** if it is clock-wise.

Angles in Standard Position

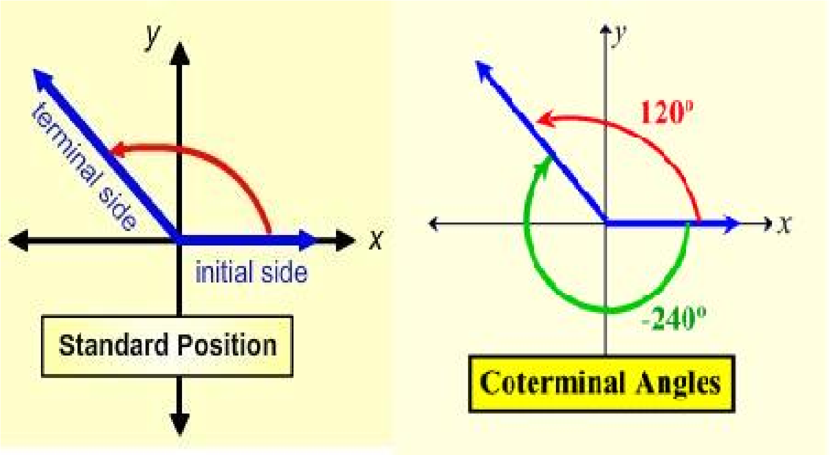


Figure: An angle in STANDARD POSITON has the $+x$-axis as its initial side. More than one angle may have the same terminal side. These are called *co-terminal*.

Degree Measure

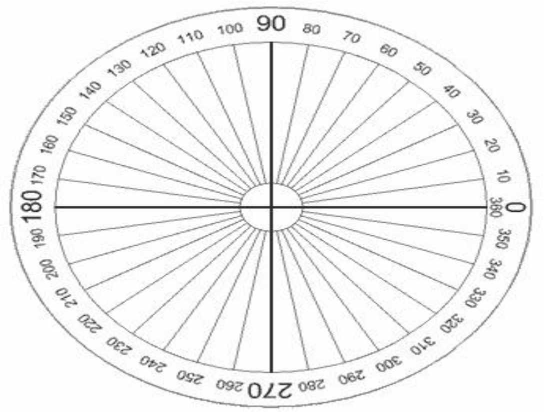


Figure: We can asign a measure to the angle between an initial and terminal side. **Degree** measure is obtained by dividing one full rotation into 360 equal parts.

Coterminal Angles

Since we can measure clockwise (neg.) or counter clockwise (pos.), and can allow for full rotations, angles in standard position may be coterminal but of different measure.

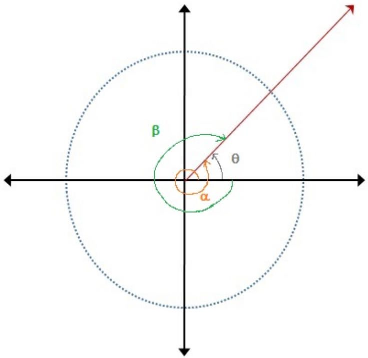


Figure: The three angles θ, α, and β have different measures but are coterminal. **Note: Coterminal angles differ by a multiple of** $360°$**.**

Example

Find two angles that are coterminal with 30°. Choose one that is negative, and one whose measure is greater than 360°.

Example

Which of the following angles is coterminal with -45°?

(a) 45°

(b) 135°

(c) 225°

(d) 315°

Supplementary Angles

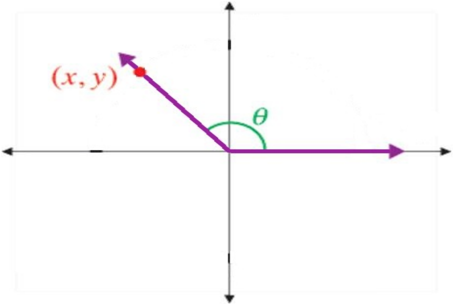
Recall that we called two actute angles whose measures sum to 90° **complementary** angles. We have a term for positive angles whose measures sum to 180°.

Definition: Two positive angles whose measures sum to 180° are called **supplementary** angles.

Example: Find the complementary and the supplementary angles for 38°.

Trigonometric Functions of any Angle

We wish to extend the definitions of the six trigonometric functions to angles that are not necessarily acute. To start, consider an angle in standard position, and choose a point (x, y) on the terminal side.



Trigonometric Function of any Angle

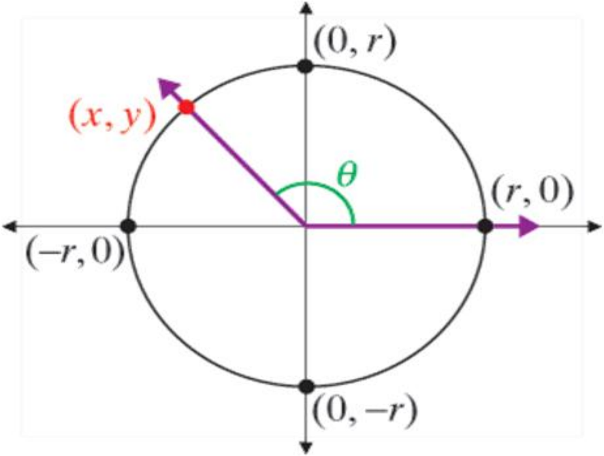


Figure: An angle in standard position determined by a point (x, y). Any such point lives on a circle in the plane centered at the origin having radius $r = \sqrt{x^2 + y^2}$

Trigonometric Function of any Angle

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} \quad (\text{for } x \neq 0)$$

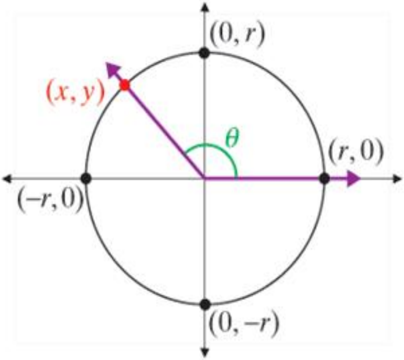


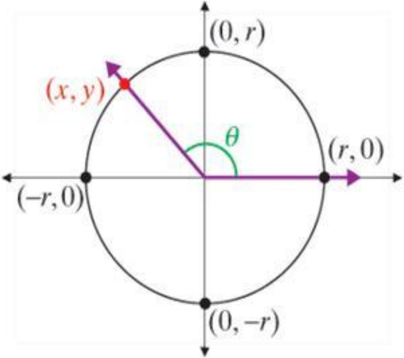
Figure: The definitions for the sine, cosine and tangent of any angle θ are given in terms of x, y, and r.

Trigonometric Function of any Angle

$$\csc\theta = \frac{r}{y} \quad (\text{for } y \neq 0)$$

$$\sec\theta = \frac{r}{x} \quad (\text{for } x \neq 0)$$

$$\cot\theta = \frac{x}{y} \quad (\text{for } y \neq 0)$$



Trigonometric Function of any Angle (Unit Circle Case)

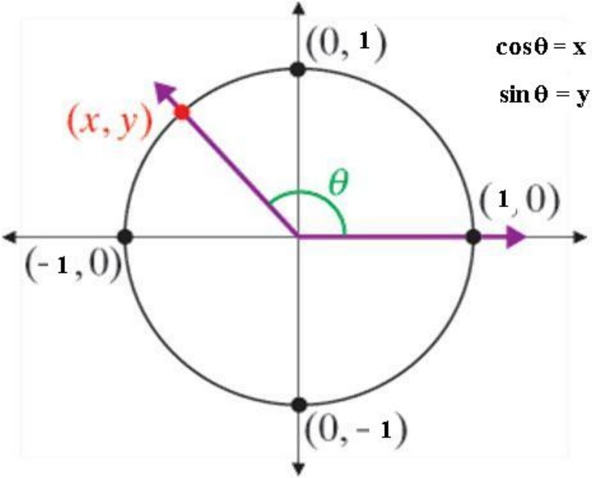


Figure: A point on the unit circle, $r = 1$, has coordinates $(x, y) = (\cos\theta, \sin\theta)$.

Reciprocal Identities

We have the first in a long list of **trigonometric identities**:

Reciprocal Identities: For any given θ for which both sides are defined

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \& \quad \cot\theta = \frac{1}{\tan\theta}.$$

Equivalently

$$\sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \& \quad \tan\theta = \frac{1}{\cot\theta}.$$

Comparison to Acute Angle Definitions

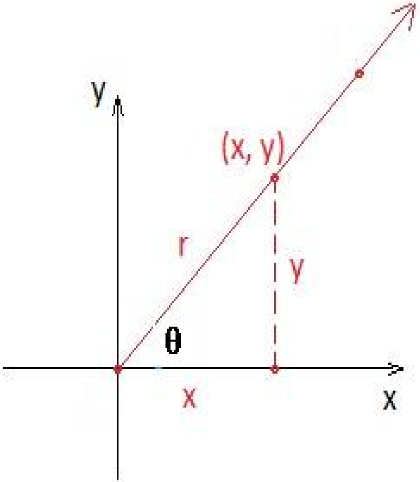


Figure: Note that the acute angle definitions still hold.

A Couple of Degenerate Triangles

Determine the six trigonometric values of 0° and 90° as possible.

Quadrantal Angles

The angles 0° and 90° both have the property that when put in standard position, the terminal side is concurrent with one of the coordinate axes. Such angles are called **quadrantal angles.** Some other quadrantal angles include

$$180^\circ, \quad 270^\circ, \quad -90^\circ, \quad \text{and} \quad 360^\circ$$

A Useful Table of Trigonometric Values

θ°	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

Quadrants & Signs

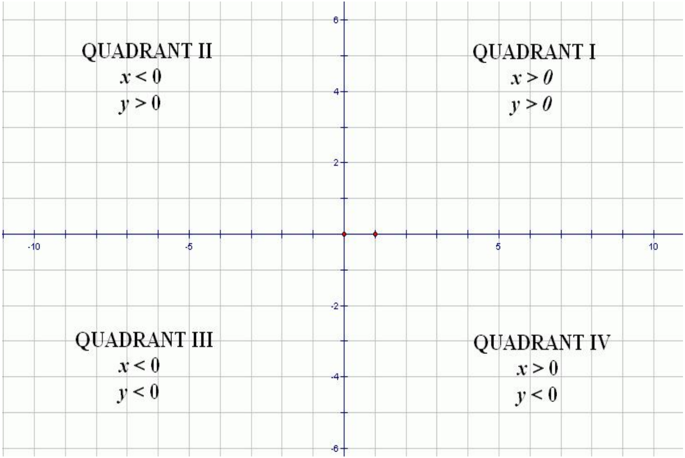
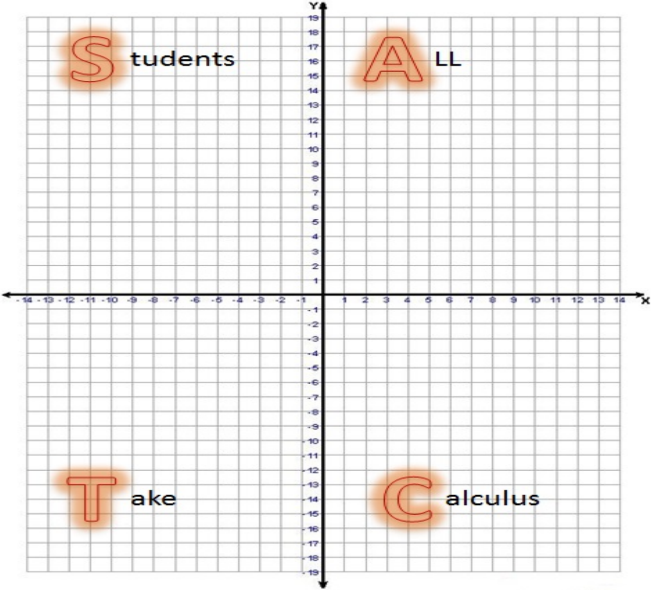


Figure: The trigonometric values for a general angle may be positive, negative, zero, or undefined. The signs are determined by the signs of the x and y values. **Note that $r > 0$ by definition.**

Quadrants & Signs of Trig Values



Example

Determine which quadrant the terminal side of θ must be in if

(a) $\sin\theta > 0$ and $\tan\theta < 0$

(b) $\sec\theta < 0$ and $\cot\theta > 0$

Question

Suppose that $\sin\theta = -0.3420$ and $\cos\theta = -0.9397$. Which of the following must be true about θ?

(a) $0^\circ < \theta < 90^\circ$

(b) $90^\circ < \theta < 180^\circ$

(c) $180^\circ < \theta < 270^\circ$

(d) $270^\circ < \theta < 360^\circ$

(e) any of the above may be true, more information is needed to determine which is true

Reference Angles

Suppose we want to find the trig values for the angle θ shown. Note that the acute angle (pink) has terminal side through (x, y), and by symmetry the terminal side of θ passes through the point $(-x, y)$ (same y and opposite sign x).

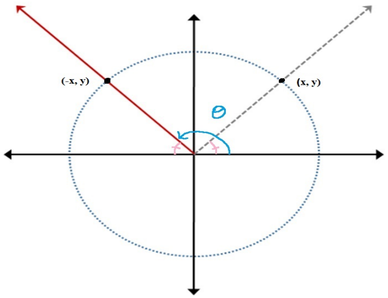
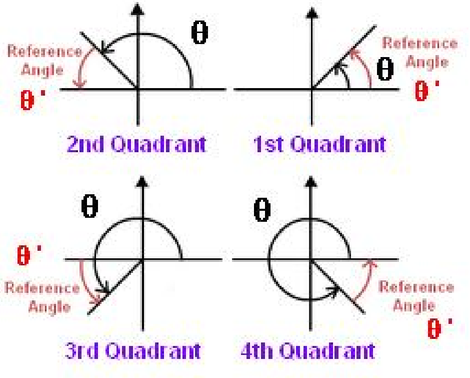


Figure: What is the connection between the trig values for θ and those for the acute angle in pink?

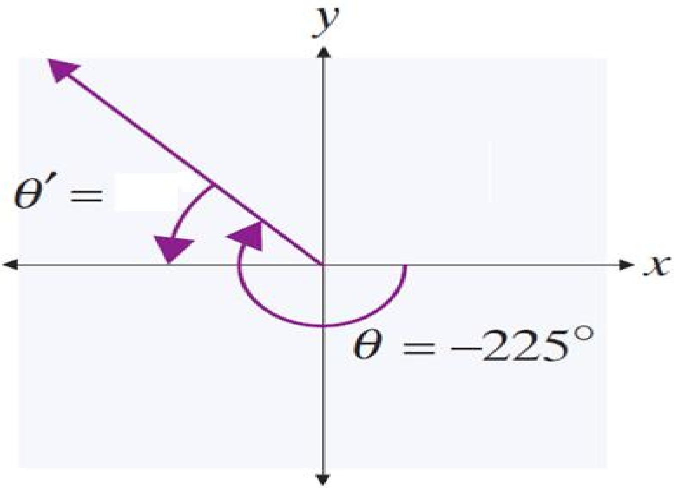
Reference Angles

Definition: Let θ be an angle in standard position. The **reference angle** θ' associated with θ is the angle of measure $0° < \theta' < 90°$ between the terminal side of θ and the *nearest* part of the *x*-axis.



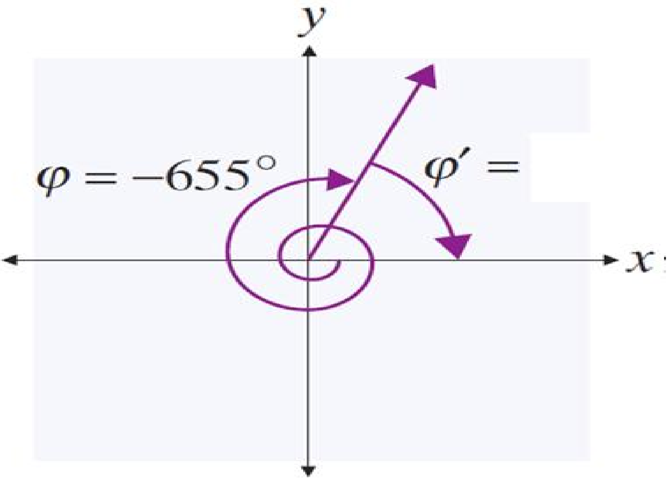
Example (a)

Determine the reference angle.



Example (b)

Determine the reference angle.



Question

The reference angle for 300° is

(a) -60°

(b) 60°

(c) -30°

(d) 30°

Theorem on Reference Angles

Theorem: If θ' is the reference angle for the angle θ, then

$$\sin\theta' = |\sin\theta|, \quad \cos\theta' = |\cos\theta| \quad \& \quad \tan\theta' = |\tan\theta|.$$

Remark 1: The analogous relationships hold for the cosecant, secant, and cotangent.

Remark 2: This means that the trigonometric values for θ can differ at most by a sign (+ or -) from the values for θ'.

Example: Using Reference Angles

Find the exact value of

(a) $\sin(135^\circ)$

(b) $\cos(210^\circ)$

Question

Suppose θ is an angle such that 270° $< \theta <$ 360° and its reference angle θ' satisfies

$$\cos\theta' = \frac{2}{3}.$$

The secant of θ,

(a) $\sec\theta = \frac{3}{2}$, and I'm certain

(b) $\sec\theta = \frac{3}{2}$, but I'm not certain

(c) $\sec\theta = -\frac{3}{2}$, and I'm certain

(d) $\sec\theta = -\frac{3}{2}$, but I'm not certain

More New Trigonometric Identities

Quotient Identities: For any given θ for which both sides are defined

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \& \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Example

Use the given information to determine the remaining trigonometric values of θ.

(a) $\sin\theta = \dfrac{1}{4}$ and $\cos\theta = -\dfrac{\sqrt{15}}{4}$

Question

Suppose we know that $\cos\theta = \dfrac{2}{\sqrt{13}}$ and $\cot\theta = -\dfrac{2}{3}$

Which of the following must be true?

(a) θ has terminal side in quadrant 4

(b) $\sin\theta = -\dfrac{3}{\sqrt{13}}$

(c) $\tan\theta = -\dfrac{3}{2}$

(d) All of the above are true.

(e) None of the above are true.