Oct. 19 Math 1190 sec. 52 Fall 2016

Section 4.3: The Mean Value Theorem

Rolle's Theorem: Let f be a function that is

- i continuous on the closed interval $[a, b]$,
- ii differentiable on the open interval (a, b), and
- iii such that $f(a) = f(b)$.

Then there exists a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem (MVT) is arguably the most significant theorem in calculus. This is even accounting for a theorem we'll discuss later called the *Fundamental Theorem of Calculus*.

Rolle's Theorem

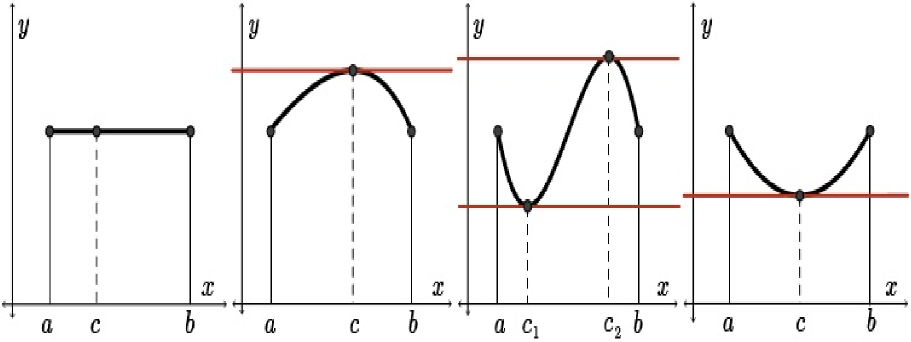
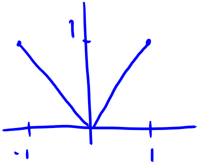


Figure: The theorem illustrated: How to find *c* is not indicated, but its existence is guaranteed.

Question

Let $f(x) = |x|$. Note that f is continuous on $[-1, 1]$ and $f(-1) = f(1) = 1$. However, **there is no c value in $(-1, 1)$ such that $f'(c) = 0$.** Which of the following is true?



(a) This contradicts Rolle's theorem.

(b) f is not really continuous on $[-1, 1]$, so Rolle's theorem doesn't apply.

(c) f is not differentiable on $(-1, 1)$, so Rolle's theorem doesn't apply.

The Mean Value Theorem

Theorem: Suppose f is a function that satisfies

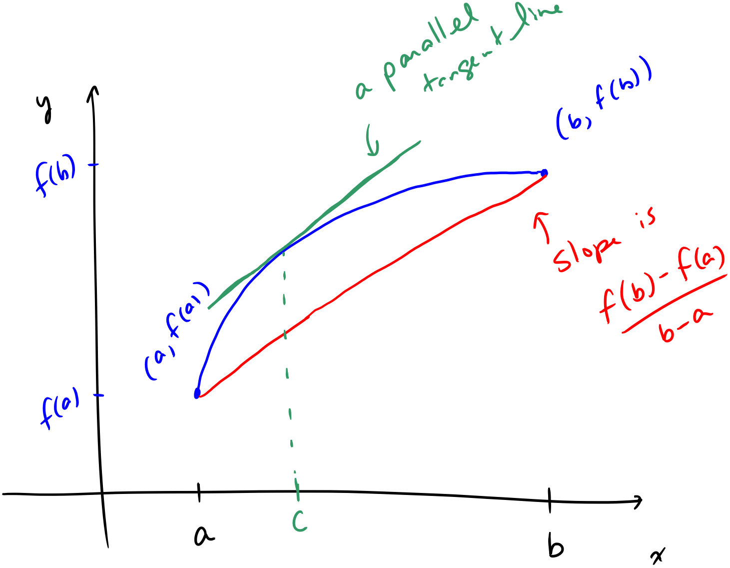
i f is continuous on the closed interval $[a, b]$, and

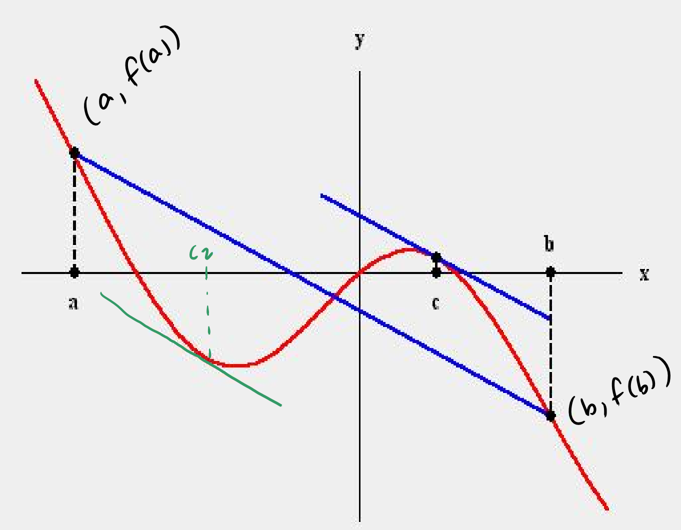
ii f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

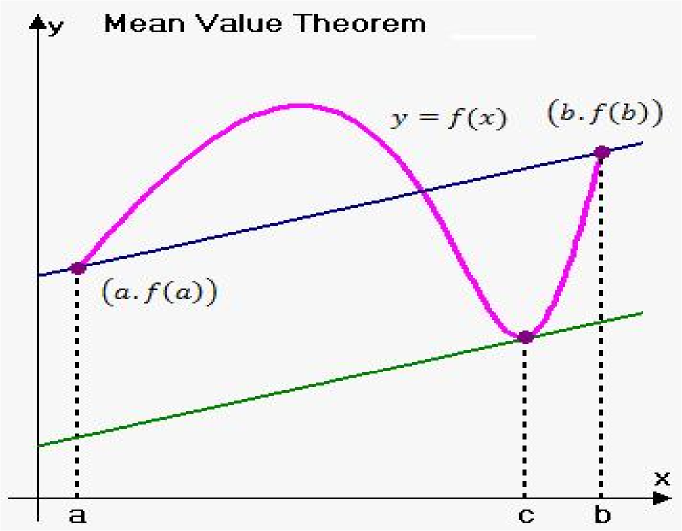
$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

this is the slope of the secant line between $(a, f(a))$ and $(b, f(b))$


a parallel tangent line
(b, f(b))
(a, f(a))
f(b)
f(a)
y
x
a
c
b
slope is
$\frac{f(b)-f(a)}{b-a}$



Figure



Figure

Figure: Celebration of the MVT in Beijing.

Example

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all values of c that satisfy the conclusion of the MVT.

$$f(x) = x^3 - 2x, \quad [0, 2]$$

As a polynomial, f is continuous on $(-\infty, \infty)$ hence on $[0,2]$. It's also differentiable on $(-\infty, \infty)$ hence on $(0,2)$.

We want to find all c in $(0,2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(0)}{2 - 0}$$

$$f'(c) = \frac{2^3 - 2\cdot 2 - (0^3 - 2\cdot 0)}{2} = \frac{8-4-0}{2} = \frac{4}{2} = 2$$

$f'(x) = 3x^2 - 2$ so we'll solve

$$f'(c) = 3c^2 - 2 = 2 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3}$$

$$c = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \quad \text{or} \quad c = -\sqrt{\frac{4}{3}} = \frac{-2}{\sqrt{3}}$$

this is not in $[0,2]$

There is one c value in $(0,2)$, $c = \frac{2}{\sqrt{3}}$.

Question

Let $f(x) = \sqrt{x-2}$, and let $[a, b] = [2, 6]$.

(1) The function f satisfies the hypotheses of the MVT. Determine

$$\frac{f(b) - f(a)}{b - a}.$$

$$\frac{f(b)-f(a)}{b-a} = \frac{\sqrt{6-2} - \sqrt{2-2}}{6-2}$$

$$= \frac{\sqrt{4} - 0}{4} = \frac{2}{4} = \frac{1}{2}$$

(a) $\dfrac{f(b) - f(a)}{b - a} = \dfrac{2}{3}$

(b) $\dfrac{f(b) - f(a)}{b - a} = \dfrac{3}{2}$

(c) $\dfrac{f(b) - f(a)}{b - a} = 2$

(d) $\dfrac{f(b) - f(a)}{b - a} = \dfrac{1}{2}$

Question

Let $f(x) = \sqrt{x-2}$, and let $[a, b] = [2, 6]$.

(2) The function f satisfies the hypotheses of the MVT. Determine

$$f'(x).$$

$f(x) = (x-2)^{1/2}$

$f'(x) = \frac{1}{2}(x-2)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x-2}}$

(a) $f'(x) = -\sqrt{x-2}$

(b) $f'(x) = \dfrac{1}{\sqrt{x-2}}$

(c) $f'(x) = \dfrac{1}{2\sqrt{x-2}}$

(d) $f'(x) = -\dfrac{1}{2\sqrt{x-2}}$

Question

$$\frac{f(b)-f(a)}{b-a} = \frac{1}{2}$$

$$f'(x) = \frac{1}{2\sqrt{x-2}}$$

Let $f(x) = \sqrt{x-2}$, and let $[a, b] = [2, 6]$.

(3) Find all c values guaranteed by the MVT for f on this interval.

(a) $c = \frac{1}{2}$

(b) $c = 3$ or $c = -3$

(c) $c = 3$ or $c = 4$

(d) $c = 3$

$$\frac{1}{2} = \frac{1}{2\sqrt{c-2}}$$

$$\frac{2\sqrt{c-2}}{2} = 1$$

$$\sqrt{c-2} = 1 \Rightarrow c-2=1$$

$$c = 3$$