5.1.4: Series Circuit Analog

Potential Drops Across Components:
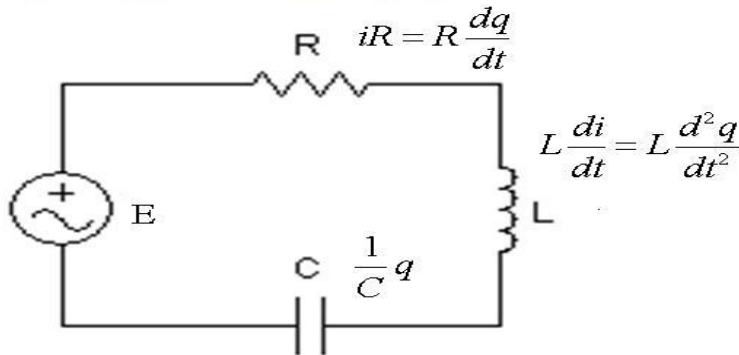

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies
$Lq'' + Rq' + \frac{1}{C}q = E(t)$.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free**. These are categorized as

overdamped if	$R^2 - 4L/C > 0$,
critically damped if	$R^2 - 4L/C = 0$,
underdamped if	$R^2 - 4L/C < 0$.

Example

An *LRC* series circuit with no applied force has an inductance of $L = 2h$ and capacitance of $C = 5 \times 10^{-3}f$. Determine the condition on the resistor such that the electrical vibrations are

$$L = 2h, \quad C = 5 \cdot 10^{-3} f$$

$$R \text{ unknown}$$

(a) Overdamped, $\quad R^2 - 1600 > 0$

$$R > 40$$

(b) Critically damped, or

$$R^2 - 1600 = 0, \quad R = 40$$

(c) Underdamped. $\quad R^2 - 1600 < 0 \quad \Rightarrow \quad 0 \le R < 40$

$$R^2 - \frac{4L}{C} = R^2 - \frac{4 \cdot 2}{5 \cdot 10^{-3}} = R^2 - 4 \cdot 2 \cdot 200 = R^2 - 1600$$

Example

An LRC-series circuit with inductance 1 h, resistance 100Ω and capacitance 0.0004 f has an applied force of 30 V. Find the charge q on the capictor if $q(0) = 0$ C and the initial current $i(0) = 2$ A. Find the maximum charge on the capacitor.

$$L q'' + R q' + \frac{1}{C} q = E \qquad q(0) = 0, \quad q'(0) = i(0) = 2$$

$$q'' + 100 q' + 2500 q = 30 \qquad q = q_c + q_p$$

Find q_c: $\quad q'' + 100 q' + 2500 q = 0$

$$m^2 + 100m + 2500 = 0 \implies (m + 50)^2 = 0$$
$$m = -50 \quad \text{repeated}$$

$$q_c = c_1 e^{-50t} + c_2 t e^{-50t}$$

Find q_p : Use Undetermined Coeff

$$q_p = A , \quad q_p' = q_p'' = 0$$

$$q_p'' + 100 q_p' + 2500 q_p = 30 \implies 2500 A = 30$$

$$A = \frac{3}{250}$$

The general solution is

$$q = c_1 e^{-50t} + c_2 t e^{-50t} + \frac{3}{250}$$

$$q(0) = 0$$

$$q'(t) = -50 C_1 e^{-50t} + C_2 e^{-50t} - 50 C_2 t e^{-50t}$$

$$q'(0) = 2$$

$$q(0) = C_1 e^0 + C_2 \cdot 0 \, e^0 + \frac{3}{250} = 0 \implies C_1 = \frac{-3}{250}$$

$$q'(0) = -50 \left(\frac{-3}{250}\right) e^0 + C_2 e^0 - 50 C_2 \cdot 0 e^0 = 2$$

$$\frac{15}{25} + C_2 = 2 \implies C_2 = \frac{50}{25} - \frac{15}{25} = \frac{35}{25} = \frac{7}{5}$$

The charge on the Capacitor is

$$q = \frac{-3}{250} e^{-50t} + \frac{7}{5} t e^{-50t} + \frac{3}{250} .$$

To find the maximum charge, we'll find all critical points and identify the absolute maximum.

$$q'(t) = \frac{150}{250} e^{-50t} + \frac{7}{5} e^{-50t} - 50 \cdot \frac{7}{5} t \, e^{-50t}$$

$$q'(t) = 0 \implies \frac{3}{5} e^{-50t} + \frac{7}{5} e^{-50t} - 70 t \, e^{-50t} = 0$$

$$e^{-50t} (2 - 70t) = 0$$

$$\implies 2 - 70t = 0 \implies t = \frac{2}{70} = \frac{1}{35}$$

We have one critical number $\frac{1}{35}$.

$$q'(0) = 2 > 0 \quad , \quad q'(1) = -68 e^{-50} < 0$$

Our local max is an absolute max since

$$q \to \frac{3}{250} \quad \text{as} \quad t \to \infty.$$

The maximum charge on the capacitor is

$$q\left(\frac{1}{35}\right) = \frac{-3}{250} e^{-\frac{50}{35}} + \frac{7}{5}\left(\frac{1}{35}\right) e^{-\frac{50}{35}} + \frac{3}{250} \approx 0.0187$$

Coulombs

Section 7.1: The Laplace Transform

If $f = f(s, t)$ is a function of two variables s and t, and we compute a definite integral **with respect to** t,

$$\int_a^b f(s, t)\, dt$$

we are left with a function of s alone.

Example: Compute the integral[1]

$$\int_0^4 (2st + s^2 - t)\, dt \;=\; \left. st^2 + s^2 t - \frac{t^2}{2} \right|_0^4$$

$$= s(4)^2 + s^2(4) - \frac{4^2}{2} - \left(s \cdot 0^2 + s^2 \cdot 0 - \frac{0^2}{2} \right)$$

$$= 16s + 4s^2 - 8 \qquad \text{a function of } s$$

---

[1] The variable s is treated like a constant when integrating with respect to t—and visa versa.

October 16, 2015   11 / 70

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that for α, β constants

$$\int_a^b K(s,t)(\alpha f(t) + \beta g(t))\,dt = \alpha \int_a^b K(s,t)f(t)\,dt + \beta \int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s, t) = e^{-st}$.