October 19 Math 2306 sec. 53 Fall 2018

Section 11: Linear Mechanical Equations

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \ge 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Pure Resonance: If the exciter frequency γ is equal to the natural frequency ω, the system experiences pure resonance. This results in a steady state with a growing (unbounded) amplitude.

Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Example

A 2 slug object is attached to a spring whose spring constant is 162 lb/ft. The system is undamped, and an external driving force of $f(t) = -4\cos(\gamma t)$ is applied. Assume that the driving force induces pure resonance. If the object starts from rest at equilibrium, determine the displacement for $t > 0$. If the spring has a maximum stretched length of 4 ft, after how many seconds will the amplitude of the oscillations exceed the maximum spring length?

$mx'' + kx = f(t)$

Note on Initial Conditions:

undamped $\Rightarrow \beta = 0$

starts from equilibrium means $x(0) = 0$,

starts from rest means $x'(0) = 0$.

$m = 2, \quad k = 162$

resonance $\Rightarrow \gamma = \omega$

Example: $m = 2$, $k = 162$, $f(t) = -4\cos(\gamma t)$

and $\gamma = \omega$

$$2x'' + 162x = -4\cos(\gamma t)$$

$$x'' + 81x = -2\cos(\gamma t) \qquad \omega^2 = 81 \Rightarrow \omega = 9,\ \gamma = 9$$

Charac. eqn $\quad r^2 + 81 = 0 \Rightarrow r = \pm 9i$

$$x_c = c_1\cos(9t) + c_2\sin(9t)$$

$$F(t) = -2\cos(9t)$$

Using undetermined coef.

$$x_p = \left(A\cos(9t) + B\sin(9t)\right)\cdot t$$

$$= At\cos(9t) + Bt\sin(9t)$$

After some computations

$$x_p'' = -81\left(At\cos(9t) + Bt\sin(9t)\right)$$
$$-18A\sin(9t) + 18B\cos(9t)$$

$$x_p'' + 81x_p = -2\cos(9t)$$

$$-18A\sin(9t)+18B\cos(9t)=-2\cos(9t)$$

$$A=0, \quad B=\frac{-1}{9}$$

So $x_p=\frac{-1}{9}t\sin(9t)$ and

$$x=c_1\cos(9t)+c_2\sin(9t)-\frac{1}{9}t\sin(9t)$$

The initial conditions were

$$x(0)=0, \quad x'(0)=0$$

Turns out, $c_1 = c_2 = 0$

The displacement $x(t) = \frac{-1}{9} t \sin(9t)$

The spring has a maximum length of 4 ft.

When does x exceed this 4 ft?

x has amplitude $A = \left|\frac{-1}{9} t\right| = \frac{t}{9}$

$$\frac{t}{9} > 4 \implies t > 36$$

The length exceeds 4 ft when $t = 36$ sec.

Section 12: LRC Series Circuits

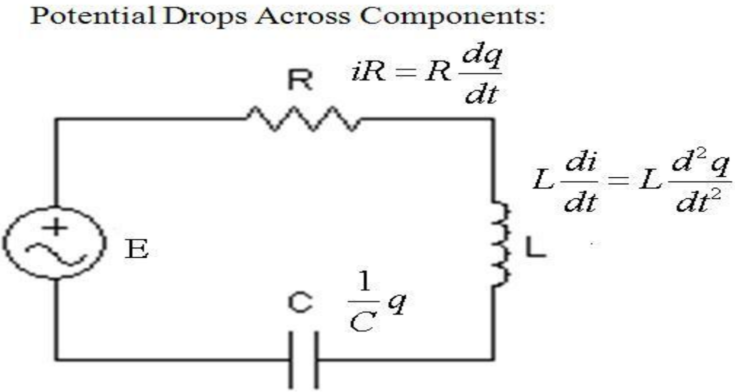

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

overdamped if	$R^2 - 4L/C > 0,$	2 real roots
critically damped if	$R^2 - 4L/C = 0,$	one repeated
underdamped if	$R^2 - 4L/C < 0.$	complex

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The ODE is

$$0.5q'' + 10q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t) \quad \Longrightarrow$$

$$q'' + 20q' + 500q = 10\cos(10t)$$

with characteristic polynomial and roots

$$r^2 + 20r + 500 = 0 \quad \Longrightarrow \quad r = -10 \pm 20i.$$

So the transient state

$$q_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t).$$

$$\frac{E(t)}{h} = 10\cos(10t)$$

$$q_p = A\cos(10t) + B\sin(10t)$$

Upon substitution into the ODE we get

$$400A + 200B = 10$$
$$-200A + 400B = 0$$

$$A = \frac{2}{100}, \quad B = \frac{1}{100}$$

The steady state current is

$$y_p = 0.02\cos(10t) + 0.01\sin(10t)$$