October 19 Math 2306 sec 54 Fall 2015

5.1.4: Series Circuit Analog

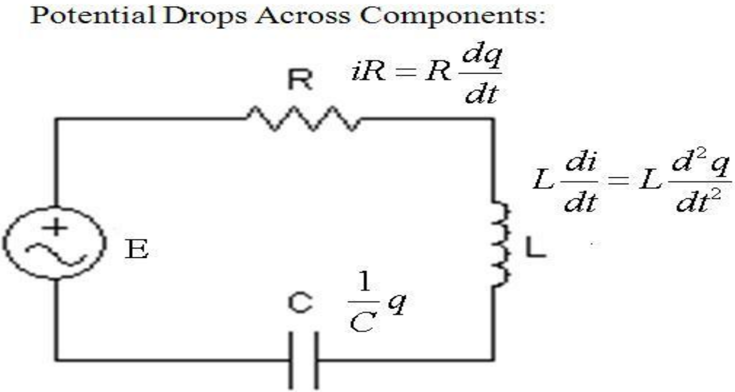

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{lrcl} \textbf{overdamped} \text{ if} & R^2 - 4L/C & > & 0, \\ \textbf{critically damped} \text{ if} & R^2 - 4L/C & = & 0, \\ \textbf{underdamped} \text{ if} & R^2 - 4L/C & < & 0. \end{array}$$

Example

An *LRC* series circuit with no applied force has an inductance of $L = 2$h and capacitance of $C = 5 \times 10^{-3}$f. Determine the condition on the resistor such that the electrical vibrations are

$L = 2h$, $C = 5 \cdot 10^{-3} f$

R - unknown

(a) Overdamped, $R^2 - 1600 > 0 \Rightarrow R^2 > 1600$

$R > 40\,\Omega$

(b) Critically damped, or

$5 \cdot 10^{-3} = \frac{5}{1000}$

$R^2 - 1600 = 0 \Rightarrow R = 40\,\Omega$

(c) Underdamped. $R^2 - 1600 < 0 \Rightarrow 0 \le R < 40\,\Omega$

$$R^2 - \frac{4L}{C} = R^2 - \frac{4 \cdot 2}{5 \cdot 10^{-3}} = R^2 - 4 \cdot 2 \cdot 200 = R^2 - 1600$$

Example

An LRC-series circuit with inductance 1 h, resistance 100Ω and capacitance 0.0004 f has an applied force of 30 V. Find the charge q on the capictor if $q(0) = 0$ C and the initial current $i(0) = 2$ A. Find the maximum charge on the capacitor.

$$Lq'' + Rq' + \frac{1}{C}q = E \qquad q'' + 100q' + \frac{1}{\frac{4}{10000}}q = 30$$

$$q(0) = 0, \quad q'(0) = 2$$

$$q'' + 100q' + 2500q = 30 \qquad q = q_c + q_p$$

Find q_c: $\quad q'' + 100q' + 2500q = 0 \qquad m^2 + 100m + 2500 = 0$

$$(m+50)^2 = 0$$
$$m = -50$$

$$q_c = C_1 e^{-50t} + C_2 t e^{-50t}$$

Find q_p : Undetermined Coef

$$q_p = A, \quad q_p' = q_p'' = 0$$

$$q_p'' + 100 q_p' + 2500 q_p = 30 \implies 2500 A = 30 \implies A = \frac{3}{250}$$

The general solution to the ODE is

$$q = C_1 e^{-50t} + C_2 t e^{-50t} + \frac{3}{250}$$

Apply

$q(0) = 0$

$q'(0) = 2$

$$q'(t) = -50c_1 e^{-50t} + c_2 e^{-50t} - 50 c_2 t e^{-50t}$$

$$q(0) = c_1 e^0 + c_2 \cdot 0 e^0 + \frac{3}{250} = 0$$

$$c_1 = \frac{-3}{250}$$

$$q'(0) = -50\left(\frac{-3}{250}\right)e^0 + c_2 e^0 - 50 \cdot c_2 \cdot 0 e^0 = 2$$

$$c_2 = 2 - \frac{3}{5} = \frac{10-3}{5} = \frac{7}{5}$$

The charge on the capacitor is

$$q(t) = \frac{-3}{250} e^{-50t} + \frac{7}{5} t e^{-50t} + \frac{3}{250} .$$

To find the max charge, we'll find critical point(s) and identify a max.

$$q'(t) = -50\left(\frac{-3}{250}\right)e^{-50t} + \frac{7}{5} e^{-50t} - 50\left(\frac{7}{5}\right) t e^{-50t}$$

$$= 2e^{-50t} - 70te^{-50t}$$

$$= 2e^{-50t}(1-35t)$$

$$q'(t) = 0 \quad \Rightarrow \quad 1 - 35t = 0 \quad \Rightarrow \quad t = \frac{1}{35}$$

1st derivative test $q'(0) = 2 > 0$, $q'(1) = -68e^{-50} < 0$

$q\left(\frac{1}{35}\right)$ is an absolute maximum.

The maximum charge is

$$q\left(\frac{1}{35}\right) = \frac{-3}{250}e^{-\frac{50}{35}} + \frac{7}{5}\left(\frac{1}{35}\right)e^{-\frac{50}{35}} + \frac{3}{250}$$

$$\approx 0.0187 \text{ C}$$

Section 7.1: The Laplace Transform

If $f = f(s,t)$ is a function of two variables s and t, and we compute a definite integral **with respect to** t,

$$\int_a^b f(s,t)\,dt$$

we are left with a function of s alone.

Example: Compute the integral[1]

$$\int_0^4 (2st+s^2-t)\,dt = st^2+s^2t-\frac{t^2}{2}\Big|_0^4$$

$$= s(4)^2+s^2\cdot 4-\frac{4^2}{2}-\left(s\cdot 0^2+s^2\cdot 0-\frac{0^2}{2}\right) = 16s+4s^2-8$$

a function of s

[1] The variable s is treated like a constant when integrating with respect to t—and visa versa.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that for α, β constants

$$\int_a^b K(s,t)(\alpha f(t) + \beta g(t))\,dt = \alpha \int_a^b K(s,t)f(t)\,dt + \beta \int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.