Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s, t)$ is a function of two independent variables ($s$ and $t$) defined over some rectangle in the plane $a \leq t \leq b$, $c \leq s \leq d$. If we compute an integral with respect to one of these variables, say $t$,

$$\int_{\alpha}^{\beta} G(s, t) \, dt$$

- the result is a function of the remaining variable $s$, and

- the variable $s$ is treated as a constant while integrating with respect to $t$. 
Integral Transform

An integral transform is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_{a}^{b} K(s, t)f(t) \, dt.$$ 

- The function $K$ is called the kernel of the transformation.
- The limits $a$ and $b$ may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is linear in the sense that

$$\int_{a}^{b} K(s, t)(\alpha f(t) + \beta g(t)) \, dt = \alpha \int_{a}^{b} K(s, t)f(t) \, dt + \beta \int_{a}^{b} K(s, t)g(t) \, dt.$$
The Laplace Transform

Definition: Let \( f(t) \) be defined on \([0, \infty)\). The Laplace transform of \( f \) is denoted and defined by

\[
\mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) \, dt = F(s).
\]

The domain of the transformation \( F(s) \) is the set of all \( s \) such that the integral is convergent.

Note: The kernel for the Laplace transform is \( K(s, t) = e^{-st} \).

Note 2: If we take \( s \) to be real-valued, then

\[
\lim_{t \to \infty} e^{-st} = 0 \quad \text{if } s > 0, \quad \text{and} \quad \lim_{t \to \infty} e^{-st} = \infty \quad \text{if } s < 0.
\]
Find the Laplace transform of \( f(t) = 1 \)

By definition \( L\{1\} = \int_0^\infty e^{-st} \, dt = \int_0^\infty e^{-st} \, dt \)

Case 1: If \( s = 0 \), \( e^{-st} = e^0 = 1 \)

\[
\int_0^\infty 1 \, dt = \lim_{b \to \infty} \int_0^b \, dt = \lim_{b \to \infty} \left. t \right|_0^b = \lim_{b \to \infty} (b-0) = \infty
\]

The integral diverges if \( s = 0 \). \( 0 \) is not in the domain of \( F(s) \).

Case 2: For \( s \neq 0 \),

\[
L\{1\} = \int_0^\infty e^{-st} \, dt = \lim_{b \to \infty} \int_0^b e^{-st} \, dt
\]
\[
\lim_{b \to \infty} \left. \frac{1}{s} e^{-st} \right|_0^b = \lim_{b \to \infty} \left( \frac{1}{s} e^{-sb} - \frac{1}{s} e^0 \right)
\]

\[
= 0 + \frac{1}{s} \quad \text{if } s > 0.
\]

So, \( \mathcal{L}\{1\} = \frac{1}{s} \) with domain \( s > 0 \).
Find the Laplace transform of \( f(t) = t \)

By definition \( \mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt \)

If \( s = 0 \), \( e^{-st} \cdot t = e^0 \cdot t = t \). The integral is
\( \int_0^\infty t \, dt = \infty \). Zero is not in the domain.

For \( s \neq 0 \), \( \mathcal{L}\{t^2\} = \int_0^\infty e^{-st} \cdot t^2 \, dt \)

\( \mathcal{L}\{t^2\} = \left. \frac{-1}{s} e^{-st} \cdot t \right|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} \, dt \)

By parts:
\( u = t \quad du = dt \)
\( v = \frac{1}{s} e^{-st} \quad dv = e^{-st} \, dt \)
For $s > 0$:

\[
0 - 0 + \frac{1}{s} \int_0^\infty e^{-st} \, dt
\]

\[
= \frac{1}{s} \mathcal{L}\{1\} = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}
\]

\[
\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s > 0
\]
A piecewise defined function

Find the Laplace transform of \( f \) defined by

\[
 f(t) = \begin{cases} 
 2t, & 0 \leq t < 10 \\
 0, & t \geq 10
\end{cases}
\]

By definition, \( \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt \)

\[
 = \int_0^{10} e^{-st} (2t) \, dt + \int_{10}^{\infty} e^{-st} (0) \, dt
\]

\[
 = \int_0^{10} 2te^{-st} \, dt
\]

\[
 \text{If } s = 0, \quad \int_0^{10} 2t \, dt = t^2 \bigg|_0^{10} = 100 - 0 = 100
\]

\[
 \text{If } s \neq 0, \quad \int_0^{10} te^{-st} \, dt
\]

\[
 \int_0^{10} \frac{1}{s} e^{-st} \, dt
\]
\[ 2 \left[ \left. \frac{-1}{5} e^{-st} t \right|_0^1 + \frac{1}{5} \int_0^1 e^{-st} \, dt \right] \]

\[ = 2 \left[ \left. \frac{-1}{5} e^{-s1} - \frac{1}{5} e^{0} 0 \right] + \frac{1}{5} \left( \left. \frac{-1}{5} e^{-st} \right|_0^1 \right) \]

\[ = 2 \left( \frac{10}{5} e^{10s} - \frac{1}{5^2} \left( e^{-s10} - e^{-s0} \right) \right) \]

\[ = -\frac{20}{5} e^{10s} - \frac{2}{5^2} e^{-10s} + \frac{2}{5^2} \]

If \( F(s) = \mathcal{L}\{f(k)\} \), then
\[ F(s) = \begin{cases} 
100, & s = 0 \\
\frac{2}{s^2} - \frac{2}{s^2} e^{-10s} - \frac{20}{s} e^{-10s}, & s \neq 0
\end{cases} \]
The Laplace Transform is a Linear Transformation

Some basic results include:

1. \( \mathcal{L} \{ \alpha f(t) + \beta g(t) \} = \alpha F(s) + \beta G(s) \)
2. \( \mathcal{L} \{ 1 \} = \frac{1}{s}, \quad s > 0 \)
3. \( \mathcal{L} \{ t^n \} = \frac{n!}{s^{n+1}}, \quad s > 0 \) for \( n = 1, 2, \ldots \)
4. \( \mathcal{L} \{ e^{at} \} = \frac{1}{s-a}, \quad s > a \)
5. \( \mathcal{L} \{ \cos kt \} = \frac{s}{s^2+k^2}, \quad s > 0 \)
6. \( \mathcal{L} \{ \sin kt \} = \frac{k}{s^2+k^2}, \quad s > 0 \)
Examples: Evaluate

(a) \( f(t) = \cos(\pi t) \)

\[ \mathcal{L} \left\{ \cos(\pi t) \right\} = \frac{s}{s^2 + \pi^2}, \ s > 0 \]
Examples: Evaluate

(b) \( f(t) = 2t^4 - e^{-5t} + 3 \)

\[ \mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2 \mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3 \mathcal{L}\{1\} \]

\[ = 2 \frac{4!}{s^{4+1}} - \frac{1}{s^{-(-5)}} + 3 \frac{1}{s} \]

\[ = \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s} , \quad s > 0 \]
Examples: Evaluate

(c) \( f(t) = (2-t)^2 = 4 - 4t + t^2 \)

\[
\mathcal{L}\{ f(t)^2 \} = 4 \mathcal{L}\{1\} - 4 \mathcal{L}\{t\} + \mathcal{L}\{t^2\} \\
= 4 \cdot \frac{1}{s} - 4 \cdot \frac{1}{s^2} + \frac{2!}{s^{2+1}} \\
= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, \quad s > 0
\]
Definition: Let $c > 0$. A function $f$ defined on $[0, \infty)$ is said to be of exponential order $c$ provided there exists positive constants $M$ and $T$ such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function $f$ is said to be piecewise continuous on an interval $[a, b]$ if $f$ has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.
Sufficient Conditions for Existence of $\mathcal{L}\{f(t)\}$

**Theorem:** If $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for some $c > 0$, then $f$ has a Laplace transform for $s > c$.

An example of a function that is NOT of exponential order for any $c$ is $f(t) = e^{t^2}$. Note that

$$f(t) = e^{t^2} = (e^t)^t \implies |f(t)| > e^{ct} \text{ whenever } t > c.$$

This is a function that doesn’t have a Laplace transform. We won’t be dealing with this type of function here.
Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$?

If so, we’ll use the following notation

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathcal{L}\{f(t)\} = F(s).$$

We’ll call $f(t)$ an *inverse Laplace transform* of $F(s)$.
A Table of Inverse Laplace Transforms

- \( \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} = 1 \)

- \( \mathcal{L}^{-1} \left\{ \frac{n!}{s^{n+1}} \right\} = t^n, \text{ for } n = 1, 2, \ldots \)

- \( \mathcal{L}^{-1} \left\{ \frac{1}{s-a} \right\} = e^{at} \)

- \( \mathcal{L}^{-1} \left\{ \frac{s}{s^2+k^2} \right\} = \cos kt \)

- \( \mathcal{L}^{-1} \left\{ \frac{k}{s^2+k^2} \right\} = \sin kt \)

The inverse Laplace transform is also linear so that

\[
\mathcal{L}^{-1} \{ \alpha F(s) + \beta G(s) \} = \alpha f(t) + \beta g(t)
\]
Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets \{\} **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

(a) \( \mathcal{L}^{-1}\left\{ \frac{1}{s^7} \right\} \)

Note \( \frac{1}{s^7} = \frac{1}{s^{6+1}} \cdot \frac{6!}{6!} = \frac{1}{6!} \cdot \frac{6!}{s^{6+1}} \)

So \( \mathcal{L}^{-1}\left\{ \frac{1}{s^7} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{6!} \cdot \frac{6!}{s^{6+1}} \right\} = \frac{1}{6!} \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s^{6+1}} \right\} = \frac{1}{6!} \cdot t^6 \)

Use \( \mathcal{L}^{-1}\left\{ \frac{n!}{s^{n+1}} \right\} = t^n \)
Example: Evaluate

$$L^{-1}\left\{ \frac{s + 1}{s^2 + 9} \right\} = L^{-1}\left\{ \frac{s}{s^2 + 9} \right\} + L^{-1}\left\{ \frac{1}{s^2 + 9} \right\}$$

$$= \frac{s}{s^2 + 3^2} + \frac{1}{3} \frac{3}{s^2 + 3^2}$$

$$L^{-1}\left\{ \frac{s + 1}{s^2 + 9} \right\} = \cos(3t) + \frac{1}{3} \sin(3t)$$