October 19 Math 2306 sec. 56 Fall 2017

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_\alpha^\beta G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s, t) = e^{-st}$.

Note 2: If we take s to be real-valued, then

$$\lim_{t\to\infty} e^{-st} = 0 \quad \text{if } s > 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-st} = \infty \quad \text{if } s < 0.$$

Find the Laplace transform of $f(t) = 1$

By definition $\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt = \int_0^\infty e^{-st} \, dt$

Case 1: If $s=0$, $e^{-st} = e^0 = 1$

$$\int_0^\infty 1 \, dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t \Big|_0^b = \lim_{b\to\infty} (b-0) = \infty$$

The integral diverges if $s=0$. 0 is not in the domain of $F(s)$.

Case 2: For $s \neq 0$,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt = \lim_{b\to\infty} \int_0^b e^{-st} \, dt$$

$$= \lim_{b\to\infty} \frac{1}{-s} e^{-st}\Big|_0^b = \lim_{b\to\infty}\left(\frac{-1}{s}e^{-sb} - \frac{-1}{s}e^{0}\right)$$

$$= 0 + \frac{1}{s} \quad \text{if} \quad s>0.$$

So $\mathscr{L}\{1\} = \frac{1}{s}$ with domain $s>0$.

Find the Laplace transform of $f(t) = t$

By definition $\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$

If $s = 0$, $e^{-st} \cdot t = e^{0} t = t$. The integral is

$\int_0^\infty t \, dt = \infty$. Zero is not in the domain.

By parts

$u = t \quad du = dt$

$v = \frac{-1}{s} e^{-st} \quad dv = e^{-st} dt$

For $s \neq 0$, $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t \, dt$

$= \frac{-1}{s} e^{-st} t \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} \, dt$

For $s>0$

$$= 0 - 0 + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}$$

this is $\mathcal{L}\{1\}$

$$= \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty \underbrace{e^{-st}(0)}_{=0}\,dt$$

$$= \int_0^{10} 2e^{-st}t\,dt$$

If $s=0$, $\int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100-0 = 100$

If $s \neq 0$, $\int_0^{10} 2te^{-st}\,dt$

parts

$u = t \qquad du = dt$

$v = \frac{-1}{s}e^{-st} \qquad dv = e^{-st}\,dt$

$$= 2\left[\frac{-1}{s}e^{-st}\cdot t\Big|_0^{10} + \frac{1}{s}\int_0^{10} e^{-st}\,dt\right]$$

$$= 2\left[\frac{-1}{s}e^{-s\cdot 10}\cdot 10 - \frac{-1}{s}e^{0}\cdot 0 + \frac{1}{s}\left(\frac{-1}{s}e^{-st}\Big|_0^{10}\right.\right.$$

$$= 2\left(\frac{-10}{s}e^{-10s} - \frac{1}{s^2}\left(e^{-s\cdot 10} - e^{-s\cdot 0}\right)\right)$$

$$= \frac{-20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s} + \frac{2}{s^2}$$

If $F(s) = \mathcal{L}\{f(t)\}$, then

$$F(s) = \begin{cases} 100, & s = 0 \\ \dfrac{2}{s^2} - \dfrac{2}{s^2}e^{-10s} - \dfrac{20}{s}e^{-10s}, & s \neq 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \quad s>0$$

(a) $f(t) = \cos(\pi t)$

$$\mathscr{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}$$

Examples: Evaluate

$\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$s>0$ $\quad s>0$ $\quad s>a$

(b) $\quad f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\frac{4!}{s^{4+1}} - \frac{1}{s-(-5)} + 3\frac{1}{s}$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}, \quad s>0$$

Examples: Evaluate

$\mathcal{L}\{1\} = \frac{1}{s}$ $\quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$s > 0$

(c) $f(t) = (2-t)^2$ $= 4 - 4t + t^2$

$$\mathcal{L}\{f(t)\} = 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4 \cdot \frac{1}{s} - 4\frac{1!}{s^{1+1}} + \frac{2!}{s^{2+1}}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, \quad s > 0$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

An example of a function that is NOT of exponential order for any c is $f(t) = e^{t^2}$. Note that

$$f(t) = e^{t^2} = \left(e^t\right)^t \quad \Longrightarrow \quad |f(t)| > e^{ct} \quad \text{whenever} \quad t > c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

Use $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

Note $\frac{1}{s^7} = \frac{1}{s^{6+1}} \cdot \frac{6!}{6!} = \frac{1}{6!} \frac{6!}{s^{6+1}}$

So $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!} \frac{6!}{s^{6+1}}\right\} = \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^{6+1}}\right\} = \frac{1}{6!} t^6$

Example: Evaluate

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$, $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$\frac{s+1}{s^2+9} = \frac{s}{s^2+9} + \frac{1}{s^2+9} = \frac{s}{s^2+3^2} + \frac{1}{s^2+3^2}$$

$$= \frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}$$

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \cos(3t) + \frac{1}{3}\sin(3t)$$