Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s, t)$ is a function of two independent variables $(s$ and $t)$ defined over some rectangle in the plane $a \leq t \leq b$, $c \leq s \leq d$. If we compute an integral with respect to one of these variables, say $t$,

$$
\int_{\alpha}^{\beta} G(s, t) \, dt
$$

- the result is a function of the remaining variable $s$, and
- the variable $s$ is treated as a constant while integrating with respect to $t$. 

Integral Transform

An integral transform is a mapping that assigns to a function \( f(t) \) another function \( F(s) \) via an integral of the form

\[
\int_{a}^{b} K(s, t)f(t) \, dt.
\]

- The function \( K \) is called the kernel of the transformation.
- The limits \( a \) and \( b \) may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is linear in the sense that

\[
\int_{a}^{b} K(s, t)(\alpha f(t) + \beta g(t)) \, dt = \alpha \int_{a}^{b} K(s, t)f(t) \, dt + \beta \int_{a}^{b} K(s, t)g(t) \, dt.
\]
The Laplace Transform

**Definition:** Let \( f(t) \) be defined on \([0, \infty)\). The Laplace transform of \( f \) is denoted and defined by

\[
\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \, dt = F(s).
\]

The domain of the transformation \( F(s) \) is the set of all \( s \) such that the integral is convergent.

**Note:** The kernel for the Laplace transform is \( K(s, t) = e^{-st} \).

**Note 2:** If we take \( s \) to be real-valued, then

\[
\lim_{t \to \infty} e^{-st} = 0 \quad \text{if } s > 0, \quad \text{and} \quad \lim_{t \to \infty} e^{-st} = \infty \quad \text{if } s < 0.
\]
Find the Laplace transform of \( f(t) = 1 \)

By definition \( \mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt = \int_0^\infty e^{-st} \, dt \)

If \( s = 0 \), \( e^{-st} = e^0 = 1 \) the integral is

\[
\int_0^\infty dt = \lim_{b \to \infty} \int_0^b dt = \lim_{b \to \infty} b \bigg|_0^b = \lim_{b \to \infty} (b - 0) = \infty
\]

The integral is divergent when \( s = 0 \). Zero is not in the domain of \( \mathcal{L}\{1\} \).

For \( s \neq 0 \) \( \mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt = \lim_{b \to \infty} \int_0^b e^{-st} \, dt \)
\[ \lim_{b \to \infty} \frac{-1}{s} e^{-sb} \bigg|_b^0 \]

\[ = \lim_{b \to \infty} \left( \frac{-1}{s} e^{-sb} - \frac{-1}{s} e^0 \right) \quad \text{for } s > 0 \]

\[ = 0 + \frac{1}{s} \]

So \[ g\{1\} = \frac{1}{s} \] with domain \( s > 0 \).
Find the Laplace transform of \( f(t) = t \)

By definition, \( \mathcal{L}\{t\} = \int_{0}^{\infty} e^{-st} \, t \, dt \)

If \( s = 0 \), \( e^{st} \, t = t \). The integral is \( \int_{0}^{\infty} t \, dt = \infty \).

Zero is not in the domain of \( \mathcal{L}\{t\} \).

For \( s \neq 0 \), \( \mathcal{L}\{t\} = \int_{0}^{\infty} e^{-st} \, t \, dt \)

\[ = \frac{-1}{s} \left| e^{-st} \right|_{0}^{\infty} + \frac{1}{s} \int_{0}^{\infty} e^{-st} \, dt \]

\[ = \frac{1}{s^2} + \frac{1}{s} \int_{0}^{\infty} e^{-st} \, dt \]

\[ = \frac{1}{s^2} + \frac{1}{s} \left( \frac{1}{s} \right) \]

\[ = \frac{1}{s^2} + \frac{1}{s^2} \]

\[ = \frac{1}{s} \]

for \( s > 0 \)
\begin{align*}
&= 0 - 0 + \frac{1}{s} \int_0^\infty e^{-st} dt \\
&= \frac{1}{s} \left[ \frac{1}{s} \right] = \frac{1}{s^2} \\
&= \frac{1}{s^2} \\
L \{ t^2 \} &= \frac{1}{s^2} \quad \text{for} \quad s > 0.
\end{align*}
A piecewise defined function

Find the Laplace transform of \( f \) defined by

\[
f(t) = \begin{cases} 
2t, & 0 \leq t < 10 \\
0, & t \geq 10 
\end{cases}
\]

By definition, \( \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt \)

\[
= \int_0^{10} e^{-st} (2t) \, dt + \int_{10}^{\infty} e^{-st} (0) \, dt
\]

When \( s = 0 \), we get

\[
\int_0^{10} 2t \, dt = (t^2) \bigg|_0^{10} = 100
\]

When \( s \neq 0 \),

\[
\int_0^{10} e^{-st} (2t) \, dt
\]

Int by parts:

\[
u = t, \quad dv = e^{-st} \, dt
\]

\[
u = \frac{-1}{s} e^{-st}, \quad dv = e^{-st} \, dt
\]
\[ 2 \left( -\frac{1}{s} e^{-s t} \bigg|_0^10 + \frac{1}{s} \int_0^{10} e^{-s t} \, dt \right) \]

\[ = 2 \left( \left( -\frac{1}{s} \cdot 10 e^{-s \cdot 10} - 0 \right) + \frac{1}{s} \left( \frac{-1}{s} e^{-s t} \bigg|_0^{10} \right) \right) \]

\[ = 2 \left( -\frac{10}{s} e^{-10s} - \frac{1}{s^2} \left( e^{-10s} - e^{0} \right) \right) \]

\[ = -\frac{20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s} + \frac{2}{s^2} \]

\[ \mathcal{L}\{f(t)\} = \begin{cases} 
20, & s = 0 \\
\frac{2}{s^2} - \frac{2}{s^2} e^{-10s} - \frac{20}{s} e^{-st}, & s \neq 0
\end{cases} \]
The Laplace Transform is a Linear Transformation

Some basic results include:

- \[ \mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s) \]
- \[ \mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0 \]
- \[ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0 \text{ for } n = 1, 2, \ldots \]
- \[ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a \]
- \[ \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0 \]
- \[ \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0 \]
Examples: Evaluate

(a) \( f(t) = \cos(\pi t) \)

\[
\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2 + \pi^2}, \quad s > 0
\]
Examples: Evaluate

(b) \[ f(t) = 2t^4 - e^{-5t} + 3 \]

\[
\mathcal{L}\{f(t)\} = \mathcal{L}\{2t^4\} - \mathcal{L}\{e^{-5t}\} + 3 \mathcal{L}\{1\}
\]

\[
= 2 \cdot \frac{4!}{s^{4+1}} - \frac{1}{s^{s-(-5)}} + 3 \cdot \frac{1}{s}
\]

\[
= \frac{48}{s^5} - \frac{1}{s + 5} + \frac{3}{s}, \quad s > 0
\]
Examples: Evaluate

(c) \( f(t) = (2 - t)^2 = 4 - 4t + t^2 \)

\[
\mathcal{L}\{r(t)^2\} = \mathcal{L}\{1\}^2 - \mathcal{L}\{t\} + \mathcal{L}\{t^2\}
\]

\[
= \left( \frac{1}{s} \right)^2 - \frac{1}{s} \cdot \frac{1}{s} + \frac{2!}{s^{2+1}}
\]

\[
= \frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^3}
\]
Sufficient Conditions for Existence of $\mathcal{L}\{f(t)\}$

**Definition:** Let $c > 0$. A function $f$ defined on $[0, \infty)$ is said to be of exponential order $c$ provided there exists positive constants $M$ and $T$ such that $|f(t)| < Me^{ct}$ for all $t > T$.

**Definition:** A function $f$ is said to be piecewise continuous on an interval $[a, b]$ if $f$ has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.
Sufficient Conditions for Existence of $L \{ f(t) \}$

**Theorem:** If $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for some $c > 0$, then $f$ has a Laplace transform for $s > c$.

An example of a function that is NOT of exponential order for any $c$ is $f(t) = e^{t^2}$. Note that

\[ f(t) = e^{t^2} = (e^t)^t \implies |f(t)| > e^{ct} \text{ whenever } t > c. \]

This is a function that doesn’t have a Laplace transform. We won’t be dealing with this type of function here.
Section 14: Inverse Laplace Transforms

Now we wish to go backwards: Given $F(s)$ can we find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$?

If so, we’ll use the following notation

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathcal{L}\{f(t)\} = F(s).$$

We’ll call $f(t)$ an inverse Laplace transform of $F(s)$. 
A Table of Inverse Laplace Transforms

- \( \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = 1 \)

- \( \mathcal{L}^{-1}\left\{ \frac{n!}{s^{n+1}} \right\} = t^n, \text{ for } n = 1, 2, \ldots \)

- \( \mathcal{L}^{-1}\left\{ \frac{1}{s-a} \right\} = e^{at} \)

- \( \mathcal{L}^{-1}\left\{ \frac{s}{s^2+k^2} \right\} = \cos kt \)

- \( \mathcal{L}^{-1}\left\{ \frac{k}{s^2+k^2} \right\} = \sin kt \)

The inverse Laplace transform is also linear so that

\[
\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)
\]
Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets \( \{ \} \) EXACTLY! Algebra, including partial fraction decomposition, is often needed.

(a) \( \mathcal{L}^{-1}\left\{ \frac{1}{s^7} \right\} \)

\[ \mathcal{L}^{-1}\left\{ \frac{n!}{s^{n+1}} \right\} = t^n \]

Note: \[ \frac{1}{s^7} = \frac{1}{s^{6+1}} \cdot \frac{6!}{6!} = \frac{1}{6!} \cdot \frac{6!}{s^{6+1}} \]

\[ \mathcal{L}^{-1}\left\{ \frac{1}{s^7} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{6!} \cdot \frac{6!}{s^{6+1}} \right\} = \frac{1}{6!} \cdot \mathcal{L}^{-1}\left\{ \frac{6!}{s^{6+1}} \right\} = \frac{1}{6!} \cdot t^6 \]
Example: Evaluate

\[ \mathcal{L}^{-1}\left\{ \frac{S + 1}{S^2 + 9} \right\} = \cos(kt) , \quad \mathcal{L}^{-1}\left\{ \frac{k}{S^2 + 9} \right\} = \sin(kt) \]

(b) \[ \mathcal{L}^{-1}\left\{ \frac{S + 1}{S^2 + 9} \right\} \]

\[
\frac{S+1}{S^2+9} = \frac{S}{S^2+9} + \frac{1}{S^2+9} = \frac{S}{S^2+3^2} + \frac{1}{S^2+3^2} \\
= \frac{S}{S^2+3^2} + \frac{1}{3} \cdot \frac{3}{S^2+3^2} \\
\]

\[ \mathcal{L}^{-1}\left\{ \frac{S+1}{S^2+9} \right\} = \cos(3t) + \frac{1}{3} \sin(3t) \]