# October 19 Math 2306 sec. 57 Fall 2017

## Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables ($s$ and $t$) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say $t$,

$$\int_\alpha^\beta G(s,t)\,dt$$

- the result is a function of the remaining variable $s$, and
- the variable $s$ is treated as a constant while integrating with respect to $t$.

# Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function $K$ is called the **kernel** of the transformation.
- The limits $a$ and $b$ may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

# The Laplace Transform

**Definition:** Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of $f$ is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all $s$ such that the integral is convergent.

**Note:** The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

**Note 2:** If we take $s$ to be real-valued, then

$$\lim_{t\to\infty} e^{-st} = 0 \quad \text{if } s > 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-st} = \infty \quad \text{if } s < 0.$$

# Find the Laplace transform of $f(t) = 1$

By definition $\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \int_0^\infty e^{-st}\, dt$

If $s=0$, $e^{-st} = e^0 = 1$ the integral is

$$\int_0^\infty dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t\Big|_0^b = \lim_{b\to\infty} (b-0) = \infty$$

The integral is divergent when $s=0$. Zero is not in the domain of $\mathscr{L}\{1\}$.

For $s \neq 0$ $\mathscr{L}\{1\} = \int_0^\infty e^{-st}\, dt = \lim_{b\to\infty} \int_0^b e^{-st}\, dt$

$$= \lim_{b\to\infty} \frac{-1}{s} e^{-st} \Big|_0^b$$

$$= \lim_{b\to\infty} \left( \frac{-1}{s} e^{-sb} - \frac{-1}{s} e^{0} \right) \quad \text{for } s>0$$

$$= 0 + \frac{1}{s}$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ with domain $s>0$.

# Find the Laplace transform of $f(t) = t$

By definition $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

If $s=0$, $e^{-st}t = t$. The integral is $\int_0^\infty t\, dt = \infty$.

Zero is not in the domain of $\mathscr{L}\{t\}$.

For $s \neq 0$, $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

Int by parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st} dt$

$$= \frac{-1}{s} t e^{-st} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st}\, dt$$

for $s > 0$

$$= 0 - 0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\,\frac{1}{s} = \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad \text{for} \quad s > 0.$$

# A piecewise defined function

Find the Laplace transform of $f$ defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

By definition $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty e^{-st}\underbrace{(0)}_{=0}\,dt$$

When $s=0$, we get

$$\int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100$$

When $s \neq 0$ $\quad \int_0^{10} e^{-st}(2t)\,dt$

Int by parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\,dt$

$$= 2\left( \frac{-1}{s} t e^{-st} \Big|_0^{10} + \frac{1}{s} \int_0^{10} e^{-st}\, dt \right)$$

$$= 2\left(\left( \frac{-1}{s} \cdot 10\, e^{-s \cdot 10} - 0 \right) + \frac{1}{s}\left( \frac{-1}{s} e^{-st} \Big|_0^{10} \right)\right)$$

$$= 2\left( \frac{-10}{s} e^{-10s} - \frac{1}{s^2}\left( e^{-10s} - e^{0} \right)\right)$$

$$= \frac{-20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s} + \frac{2}{s^2}$$

$$\mathcal{L}\{f(t)\} = \begin{cases} 100, & s = 0 \\ \frac{2}{s^2} - \frac{2}{s^2} e^{-10s} - \frac{20}{s} e^{-st}, & s \neq 0 \end{cases}$$

# The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t)+\beta g(t)\} = \alpha F(s)+\beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s>0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$ for $n=1,2,\ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s>0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s>0$

# Examples: Evaluate

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \quad s>0$$

(a) $f(t) = \cos(\pi t)$

$$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}, \quad s>0.$$

# Examples: Evaluate

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$s>0 \qquad s>0 \qquad s>a$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\,\frac{4!}{s^{4+1}} - \frac{1}{s-(-5)} + 3\,\frac{1}{s}$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}, \quad s>0$$

# Examples: Evaluate

$\mathcal{L}\{1\} = \frac{1}{s}$ $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$s > 0$

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$$\mathcal{L}\{(2-t)^2\} = 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4\frac{1}{s} - 4\frac{1!}{s^{1+1}} + \frac{2!}{s^{2+1}}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, \quad s > 0$$

# Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

**Definition:** Let $c > 0$. A function $f$ defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants $M$ and $T$ such that $|f(t)| < Me^{ct}$ for all $t > T$.

**Definition:** A function $f$ is said to be *piecewise continuous* on an interval $[a, b]$ if $f$ has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

# Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

**Theorem:** If $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for some $c > 0$, then $f$ has a Laplace transform for $s > c$.

An example of a function that is NOT of exponential order for any $c$ is $f(t) = e^{t^2}$. Note that

$$f(t) = e^{t^2} = \left(e^t\right)^t \quad \Longrightarrow \quad |f(t)| > e^{ct} \quad \text{whenever} \quad t > c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

# Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

# A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

# Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\}$

Note $\dfrac{1}{s^7} = \dfrac{1}{s^{6+1}} \cdot \dfrac{6!}{6!} = \dfrac{1}{6!}\,\dfrac{6!}{s^{6+1}}$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^{6+1}}\right\} = \frac{1}{6!}\,\mathscr{L}^{-1}\left\{\frac{6!}{s^{6+1}}\right\} = \frac{1}{6!}\,t^6$$

## Example: Evaluate

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$ , $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$\frac{s+1}{s^2+9} = \frac{s}{s^2+9} + \frac{1}{s^2+9} = \frac{s}{s^2+3^2} + \frac{1}{s^2+3^2}$$

$$= \frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}$$

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \cos(3t) + \frac{1}{3}\sin(3t)$$