Oct 20 Math 2253 sec. 001 Fall 2014

Section 4.2: The Definite Integral

Definition Let f be defined on an interval $[a, b]$. Let

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $[a, b]$, and $\{x_1^*, x_2^*, \ldots, x_n^*\}$ be any set of sample points. Then the **definite integral of** f **from** a **to** b is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided this limit exists. Here, the limit is taken over all possible partitions of $[a, b]$.

Terms and Notation

- **Riemann Sum:** any sum of the form $f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$
- **Integral Symbol/Sign:** $\int$ (a stretched "S" for "sum")
- **Integrable:** If the limit does exists, f is said to be integrable on $[a, b]$
- **Limits of Integration:** a is called the lower limit of integration, and b is the upper limit of integration
- **Integrand:** the expression "$f(x)$" is called the integrand

- **Differential:** dx is called a differential, it indicates what the variable is and can be thought of as the limit of Δx (just as it is in the derivative notation "$\frac{dy}{dx}$").

- **Dummy Variable/Variable of Integration:** the variable that appears in both the integrand and in the differential. For example, if the differential is dx, the dummy variable is x; it the differential is du, the dummy variable is u

$$\int_a^b f(x)\,dx$$

Important Remarks

(1) If the integral does exist, it is a **number**. That is, it does not depend on the dummy variable of integration. The following are equivalent.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(q)\,dq$$

(2) The definite integral is a limit of Riemann Sums!

(3) If f is positive and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{ the area under the curve.}$$

What if *f* is continuous, but not always positive?

We can equate the integral with an area. Assign a positive sign to a region above the *x*-axis, and a negative sign to a region below the *x*-axis.

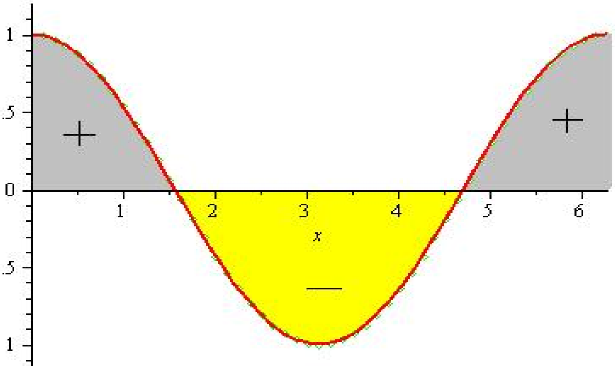
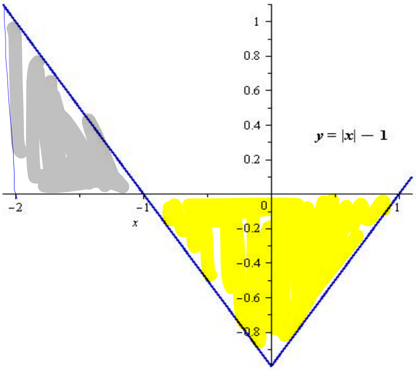


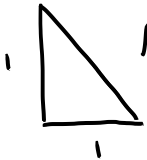
Figure: $\int_a^b f(x)\,dx =$ area of gray region $-$ area of yellow region

Example

Use area to evaluate the integral $\int_{-2}^{1}(|x|-1)\,dx$.



Gray area

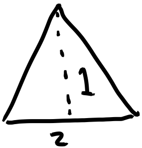


$A = \frac{1}{2}bh$

$= \frac{1}{2}\cdot 1 \cdot 1$

$= \frac{1}{2}$

Yellow area



$A = \frac{1}{2}bh = \frac{1}{2}\cdot 2 \cdot 1$

$= 1$

$$\int_{-2}^{1} (|x| - 1)\, dx = \text{Gray} - \text{Yellow}$$

$$= \frac{1}{2} - 1 = \frac{-1}{2}$$

Important Theorems:

Theorem: If f is continuous on $[a, b]$ or has only finitely many jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$

Theorem: If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad x_i = a + i\Delta x.$$

Examples:

$$\int_0^{2\pi} \cos x\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{2\pi i}{n}\right) \frac{2\pi}{n}$$

$f(x_i)$ Δx

$\Delta x = \frac{2\pi - 0}{n} = \frac{2\pi}{n}$

$x_i = 0 + i\Delta x$

$= \frac{2i\pi}{n}$

$$\int_2^4 \sqrt{t}\, dt = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{2 + \frac{2i}{n}} \left(\frac{2}{n}\right)$$

$f(x_i)$ Δx

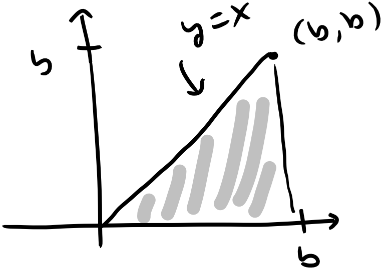
$\Delta x = \frac{4-2}{n} = \frac{2}{n}$

$x_i = 2 + i\Delta x$

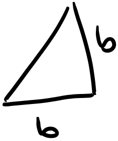
$= 2 + \frac{2i}{n}$

Show that $\int_0^b x\,dx = \frac{b^2}{2}$ by using (i) a Riemann sum[1] and (ii) geometry.

Using area



Area



$$A = \frac{1}{2} \cdot b \cdot b = \frac{1}{2} b^2$$

So

$$\int_0^b x\,dx = \frac{1}{2} b^2$$

[1] The following identity is useful

$$\sum_{i=1}^{n} i = \frac{n^2+n}{2},$$

Using a Riemann Sum:

$$\Delta x = \frac{b-0}{n} = \frac{b}{n}$$

$$x_i = 0 + i\,\Delta x = i\frac{b}{n}$$

$$f(x) = x \implies f(x_i) = i\frac{b}{n}$$

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n}\left(i\frac{b}{n}\right)\left(\frac{b}{n}\right)$$

$$= \frac{b^2}{n^2} \sum_{i=1}^{n} i$$

Using the identity $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{b^2}{n^2} \sum_{i=1}^{n} i$$

$$= \frac{b^2}{n^2} \frac{n(n+1)}{2} = \frac{b^2(n^2+n)}{2n^2}$$

Now let $n \to \infty$

$$\int_0^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \frac{b^2(n^2+n)}{2n^2}$$

$$= \lim_{n\to\infty} \frac{b^2(n^2+n)}{2n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} b^2\left(\frac{1+\frac{1}{n}}{2}\right)$$

$$= b^2\left(\frac{1+0}{2}\right) = \frac{b^2}{2}$$

So again we have

$$\int_0^b x\,dx = \frac{b^2}{2}$$

Properties of Definite Integrals

Suppose that f and g are integable on $[a, b]$ and let c be constant.

(1) $$\int_a^b c\,dx = c(b-a)$$

(2) $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

(3) $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Properties of Definite Integrals Continued...

(4) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(5) $$\int_a^a f(x)\,dx = 0$$

(6) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Properties of Definite Integrals Continued...

(7) If $f(x) \le g(x)$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

(8) And, as an immediate consequence of (7) and (1), if $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$