Oct. 21 Math 1190 sec. 51 Fall 2016

Section 4.3: The Mean Value Theorem

The Mean Value Theorem: (MVT) Suppose f is a function that satisfies

i f is continuous on the closed interval $[a, b]$, and

ii f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

Example

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all values of c that satisfy the conclusion of the MVT.

$$f(x) = x^3 - 2x, \quad [0, 2]$$

As a polynomial, f is continuous and differentiable on $(-\infty, \infty)$. So it's continuous on $[0,2]$ and differentiable on $(0,2)$.

Here, $a = 0$ and $b = 2$. We need c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f(x) = x^3 - 2x$

$f(b) = f(2) = 2^3 - 2 \cdot 2 = 4$

$f(a) = f(0) = 0^3 - 2 \cdot 0 = 0$

so $\dfrac{f(b) - f(a)}{b - a} = \dfrac{4 - 0}{2 - 0} = 2$

$f'(x) = 3x^2 - 2 \Rightarrow f'(c) = 3c^2 - 2$

$f'(c) = 3c^2 - 2 = 2$

$\Rightarrow 3c^2 = 4$

$$\Rightarrow \quad c^2 = \frac{4}{3}$$

$$\Rightarrow \quad c = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \quad \text{or} \quad c = -\sqrt{\frac{4}{3}} = \frac{-2}{\sqrt{3}}$$

ignore, it's not in $(0,2)$

There is one such c value,

$$c = \frac{2}{\sqrt{3}}.$$

Question

Let $f(x) = \sqrt{x-2}$, and let $[a, b] = [2, 6]$.

(1) The function f satisfies the hypotheses of the MVT. Determine

$$\frac{f(b) - f(a)}{b - a}.$$

(a) $\dfrac{f(b) - f(a)}{b - a} = \dfrac{2}{3}$

$$\frac{f(6) - f(2)}{6 - 2} = \frac{\sqrt{4} - \sqrt{0}}{4} = \frac{2}{4} = \frac{1}{2}$$

(b) $\dfrac{f(b) - f(a)}{b - a} = \dfrac{3}{2}$

(c) $\dfrac{f(b) - f(a)}{b - a} = 2$

(d) $\dfrac{f(b) - f(a)}{b - a} = \dfrac{1}{2}$

Question

Let $f(x) = \sqrt{x-2}$, and let $[a, b] = [2, 6]$.

(2) The function f satisfies the hypotheses of the MVT. Determine

$$f'(x).$$

$f(x) = (x-2)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x-2)^{-1/2} \cdot 1$

$= \frac{1}{2\sqrt{x-2}}$

(a) $f'(x) = -\sqrt{x-2}$

(b) $f'(x) = \dfrac{1}{\sqrt{x-2}}$

(c) $f'(x) = \dfrac{1}{2\sqrt{x-2}}$

(d) $f'(x) = -\dfrac{1}{2\sqrt{x-2}}$

Question

$$f'(x) = \frac{1}{2\sqrt{x-2}}$$

$$\frac{f(b)-f(a)}{b-a} = \frac{1}{2}$$

Let $f(x) = \sqrt{x-2}$, and let $[a, b] = [2, 6]$.

(3) Find all c values guaranteed by the MVT for f on this interval.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(a) $c = \frac{1}{2}$

(b) $c = 3$ or $c = -3$

(c) $c = 3$ or $c = 4$

(d) $c = 3$

$$\frac{1}{2\sqrt{c-2}} = \frac{1}{2}$$

$$\frac{\cancel{2}}{\cancel{2}\sqrt{c-2}} = 1$$

$$\frac{1}{\sqrt{c-2}} = 1 \Rightarrow \sqrt{c-2} = 1$$

$$c-2 = 1^2 = 1$$

$$c = 3$$

Important Consequence of the MVT

Theorem: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b). In other words,

$$f(x) = g(x) + C \quad \text{where } C \text{ is some constant.}$$

If $f'(x) = g'(x)$ for all x in (a,b),

letting $h(x) = f(x) - g(x)$ we get

$$h'(x) = f'(x) - g'(x) = 0.$$

Hence h is constant; $h(x) = C$ for some

constant C.

So $f(x) - g(x) = C \Rightarrow f(x) = g(x) + C.$

Examples

Find all possible functions $f(x)$ that satisfy the condition

(a) $f'(x) = \cos x$ on $(-\infty, \infty)$

To do this, we find one example, then add an arbitrary constant.

An example would be $\sin x$ since $\frac{d}{dx} \sin x = \cos x$.

$f(x) = \sin x + C$ for any constant C.

(b) $f'(x) = 2x$ on $(-\infty, \infty)$

One example is x^2 since $\frac{d}{dx} x^2 = 2x$.

So

$f(x) = x^2 + C$ where C is any constant.

* Note $f'(x) = \frac{d}{dx}(x^2 + C) = 2x + 0 = 2x$

Question

Find all possible functions $h(t)$ that satisfy the condition

(c) $h'(t) = \sec^2 t$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(a) $h(t) = \sec^2 t + C$, C any constant

(b) $h(t) = \tan t + 1$

(c) $h(t) = \tan t + C$, C any constant

Another Consequence of the MVT

Another significant consequence of the MVT is that it provides a test for the increasing and decreasing behavior of a differentiable function.

Theorem: Let f be differentiable on an open interval (a, b). If

- $f'(x) > 0$ on (a, b), the f is increasing on (a, b), and
- $f'(x) < 0$ on (a, b), the f is decreasing on (a, b).

Example

Determine the intervals over which f is increasing and the intervals over which it is decreasing where

$$f(x) = 2x^3 - 6x^2 - 18x + 1$$

The domain is $(-\infty, \infty)$. We need to know the sign of the 1st derivative.

$$f'(x) = 2(3x^2) - 6(2x) - 18$$

$$= 6x^2 - 12x - 18$$

$$= 6(x^2 - 2x - 3) = 6(x-3)(x+1)$$

The sign can change when $f'(x) = 0$ or where $f'(x)$ DNE.

$f'(x)$ is defined everywhere. $f'(x) = 0$

$$\Rightarrow \quad 0 = 6(x-3)(x+1) \Rightarrow \quad x = 3 \quad \text{or} \quad x = -1$$

We divide our domain by these numbers and check the sign in each interval.

sign of f'

Test: $f'(-2) = 6(-2-3)(-2+1) = 30$

$f'(0) = 6(0-3)(0+1) = -18$

$$f'(4) = 6(4-3)(4+1) = 30$$

f is increasing on $(-\infty, -1) \cup (3, \infty)$

f is decreasing on $(-1, 3)$.