Oct. 21 Math 1190 sec. 52 Fall 2016

Section 4.3: The Mean Value Theorem

The Mean Value Theorem: (MVT) Suppose f is a function that satisfies

i f is continuous on the closed interval $[a, b]$, and
ii f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

Question

$f'(c) = \frac{f(b) - f(a)}{b - a}$

Let $f(x) = x \sin x$, and let $[a, b] = \left[0, \frac{\pi}{2}\right]$.

This function is continuous on $\left[0, \frac{\pi}{2}\right]$ and differentiable on $\left(0, \frac{\pi}{2}\right)$. According to the Mean Value Theorem, there exists a number c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c)$ equals

$\frac{f(\frac{\pi}{2}) - f(0)}{\frac{\pi}{2} - 0} = \frac{\frac{\pi}{2} \sin\frac{\pi}{2} - 0 \cdot \sin 0}{\pi/2} = \frac{\pi/2}{\pi/2} = 1$

(a) $\frac{\pi}{2}$

(b) 1

(c) 0

(d) no conclusion can be drawn about the value of $f'(c)$ for any number c in $\left(0, \frac{\pi}{2}\right)$.

Important Consequence of the MVT

Theorem: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b). In other words,

$$f(x) = g(x) + C \quad \text{where } C \text{ is some constant.}$$

If $f'(x) = g'(x)$ for all x in (a,b), letting

$h(x) = f(x) - g(x)$ we have

$$h'(x) = f'(x) - g'(x) = 0 \text{ for all } x \text{ in } (a,b).$$

Then h is constant, i.e. $h(x) = C$ for some constant C. So

$$f(x) - g(x) = C \Rightarrow f(x) = g(x) + C.$$

Examples

Find all possible functions $f(x)$ that satisfy the condition

(a) $f'(x) = \cos x$ on $(-\infty, \infty)$

We find one example function, then get all possible functions by adding an arbitrary constant.

An example is $\sin x$ since $\frac{d}{dx} \sin x = \cos x$.

Thus $f(x) = \sin x + C$ for any constant C.

Check: $f'(x) = \frac{d}{dx}(\sin x + C) = \cos x + 0 = \cos x$

(b) $f'(x) = 2x$ on $(-\infty, \infty)$

x^2 is an example since $\frac{d}{dx} x^2 = 2x$

All functions are of the form

$f(x) = x^2 + C$ for any constant C

Question

Find all possible functions $h(t)$ that satisfy the condition

(c) $h'(t) = \sec^2 t$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(a) $h(t) = \sec^2 t + C$, C any constant

(b) $h(t) = \tan t + 1$

(c) $h(t) = \tan t + C$, C any constant

Another Consequence of the MVT

Another significant consequence of the MVT is that it provides a test for the increasing and decreasing behavior of a differentiable function.

Theorem: Let f be differentiable on an open interval (a, b). If

- $f'(x) > 0$ on (a, b), the f is increasing on (a, b), and
- $f'(x) < 0$ on (a, b), the f is decreasing on (a, b).

Example

Determine the intervals over which f is increasing and the intervals over which it is decreasing where

$$f(x) = 2x^3 - 6x^2 - 18x + 1$$

The domain is all reals. We need to determine where $f'(x) > 0$ and where $f'(x) < 0$.

We find where $f'(x) = 0$ or where $f'(x)$ is undefined, and determine it's sign in the intervals defined by those numbers.

$$f'(x) = 2(3x^2) - 6(2x) - 18$$

$$= 6x^2 - 12x - 18$$

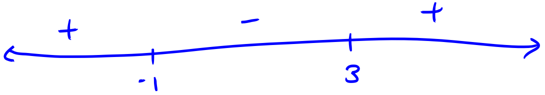
$$f'(x) = 6(x^2 - 2x - 3)$$

$$= 6(x-3)(x+1)$$

$f'(x)$ is always defined.

$$f'(x) = 0 \Rightarrow 0 = 6(x-3)(x+1)$$

$$\Rightarrow x = 3 \text{ or } x = -1$$



sign of f'

$$f'(x) = 6(x-3)(x+1)$$

Test : $f'(-2) = 6(-2-3)(-2+1) = 30$

$f'(0) = 6(0-3)(0+1) = -18$

$f'(4) = 6(4-3)(4+1) = 30$

f is increasing on $(-\infty, -1) \cup (3, \infty)$.

f is decreasing on $(-1, 3)$.

Question

Suppose that we compute the derivative of some function g and find

$$g'(x) = (2 + x)e^{x/2}.$$

Domain is $(-\infty, \infty)$

Determine the intervals over which g is increasing and over which it is decreasing.

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sign of g'

(a) g is increasing on $(-1/2, \infty)$ and decreasing on $(-\infty, -1/2)$.

(b) g is increasing on $(-2, \infty)$ and decreasing on $(-\infty, -2)$.

(c) g is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

(d) g is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.

Section 4.4: Local Extrema and Concavity

We have already seen that the first derivative f' can tell us about the behaviour of the function f—in particular, it gives information about where it is increasing or decreasing, and where it may take a local extreme value.

In this section, we'll expand on that as well as introduce information about a function that can be deduced from the nature of its second derivative.

Theorem: First derivative test for local extrema

Let f be continuous and suppose that c is a critical number of f.

- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

Note: we read from left to right as usual when looking for a sign change.

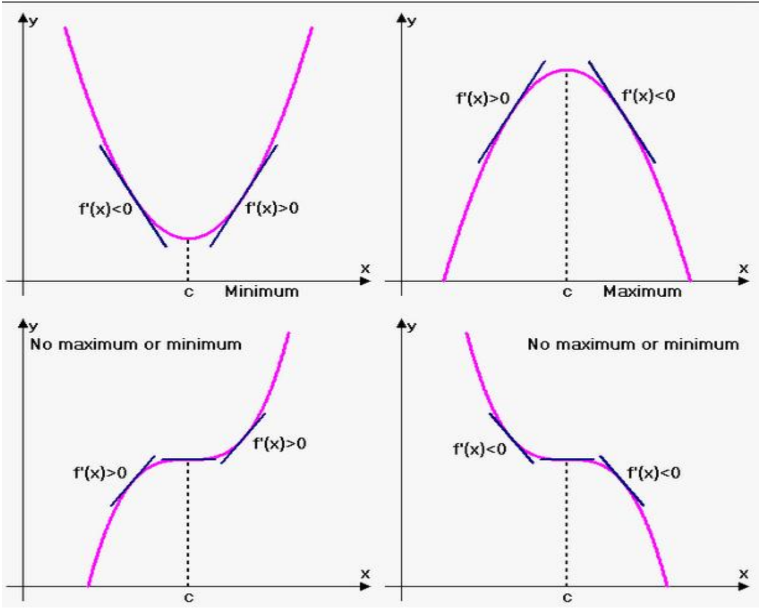


Figure: First derivative test

Example

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$f(x) = x^{1/3}(16 - x)$$

Domain is $(-\infty, \infty)$.

Find all critical numbers:

$$f(x) = 16x^{1/3} - x^{4/3}$$

$$f'(x) = 16\left(\frac{1}{3}x^{-2/3}\right) - \frac{4}{3}x^{1/3}$$

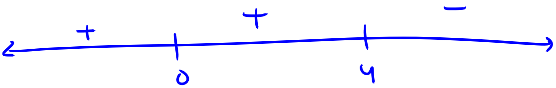
$$= \frac{16}{3x^{2/3}} - \frac{4x^{1/3}}{3} \cdot \frac{x^{2/3}}{x^{2/3}}$$

$$= \frac{16}{3x^{2/3}} - \frac{4x}{3x^{2/3}} = \frac{16-4x}{3x^{2/3}}$$

$f'(x) = 0$ if $16 - 4x = 0 \Rightarrow 4x = 16 \Rightarrow x = 4$

$f'(x)$ is undefined if $3x^{2/3} = 0 \Rightarrow x = 0$

We'll do our sign analysis



$$f'(x) = \frac{4(4-x)}{3x^{2/3}}$$

Test: $f'(-1) = \dfrac{4(4-(-1))}{3(-1)^{2/3}} = \dfrac{20}{3}$

$$f'(1) = \frac{4(4-1)}{3(1)^{2/3}} = 4$$

$$f'(8) = \frac{4(4-8)}{3(8)^{2/3}} = \frac{-16}{12} = \frac{-4}{3}$$

Classify the critical points:

f' doesn't change signs @ zero. It's not the location of a max or min.

f' changes from positive to negative @ 4.

f has a local maximum @ $x=4$.

The local maximum value is

$$f(4) = 4^{1/3}(16-4) = 12\sqrt[3]{4}$$