October 21 Math 2306 sec 51 Fall 2015

Section 7.1: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

Find the Laplace transform of $f(t) = 1$

By definition

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

If $s = 0$, then $e^{-st} = e^0 = 1$ and $\int_0^\infty dt$ which diverges

For $s \neq 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-st} \, dt = \lim_{b \to \infty} \frac{1}{-s} e^{-st} \Big|_0^b$$

$$= \frac{-1}{s} \lim_{b\to\infty} \left(e^{-sb} - e^{0} \right)$$

Diverges if $s<0$

for $s>0$

$$= \frac{-1}{s} \left(0 - 1 \right)$$

$$= \frac{1}{s} \quad \text{for} \quad s>0$$

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{with domain} \quad s>0.$$

Find the Laplace transform of $f(t) = t$

By definition

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

is $s=0$, $e^{-st} = 1$ and $\int_0^\infty t\, dt$ is divergent.

For $s \neq 0$

Int by parts

$u = t \quad du = dt$

$v = \frac{-1}{s} e^{-st} \quad dv = e^{-st}\, dt$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$$= \frac{-1}{s} t e^{-st} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st}\, dt$$

for $s>0$

$$= \frac{-1}{s}\left(0 - 0\cdot e^{0}\right) + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathscr{L}\{1\}}$$

$$= \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

So $\mathscr{L}\{t\} = \dfrac{1}{s^2}$, for $s>0$.

Find the Laplace transform of $f(t) = e^{at}$, $a \neq 0$

By definition

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

If $s = a$, $e^{-(s-a)t} = e^0 = 1$ and the integral diverges

For $s \neq a$ $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\,dt$

$$= \frac{1}{-(s-a)} e^{-(s-a)t} \Big|_0^\infty$$

For $s - a > 0$

$$= \frac{-1}{s-a}\left(0 - e^0\right) = \frac{1}{s-a}$$

Hence $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, for $s > a$

Find the Laplace transform[1] of $f(t) = \sin t$

By definition

$$\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \sin t \, dt$$

If $s = 0$, $\int_0^\infty \sin t \, dt$ diverges

For $s \neq 0$

$$\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \sin t \, dt$$

[1]The following is immensely useful!

$$\int e^{\alpha t} \sin(\beta t) \, dt = \frac{e^{\alpha t}}{\alpha^2 + \beta^2} \left(\alpha \sin(\beta t) - \beta \cos(\beta t) \right).$$

$$\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st}\sin t\,dt$$

$$= \frac{e^{-st}}{(-s)^2+1^2}\left(-s\sin t - \cos t\right)\Big|_0^\infty$$

for

$s>0$

$$= 0 - \frac{e^0}{s^2+1}\left(-s\sin 0 - \cos 0\right)$$

$$= \frac{-1}{s^2+1}(-1) = \frac{1}{s^2+1}$$

Hence

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}, \quad s>0$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

By definition

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty \underbrace{e^{-st} \cdot 0}_{=0}\,dt$$

$$= -\frac{2t}{s}e^{-st}\Big|_0^{10} + \frac{2}{s}\int_0^{10} e^{-st}\,dt$$

Int. by parts

$u = 2t \qquad du = 2\,dt$

$v = \frac{-1}{s}e^{-st} \qquad dv = e^{-st}\,dt$

$$= \frac{-2t}{s}e^{-st} - \frac{2}{s^2}e^{-st}\Big|_0^{10}$$

for $s \neq 0$

$$= \frac{-20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s} - \left(0 - \frac{2}{s^2}e^{0}\right)$$

$$= \frac{2}{s^2} - \frac{20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s}$$

for $s \neq 0$

If $s=0$,

$$\mathcal{L}\{f(t)\} = \int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100-0 = 100$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$

- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$

- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$

- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Use the Table to Evaluate $\mathscr{L}\{f(t)\}$

(a) $f(t) = 2t^2 + e^{6t}$

$$\mathscr{L}\{2t^2 + e^{6t}\} = 2\mathscr{L}\{t^2\} + \mathscr{L}\{e^{6t}\}$$

$$= 2\left(\frac{2!}{s^3}\right) + \frac{1}{s-6} = \frac{4}{s^3} + \frac{1}{s-6} \quad \text{for } s > 6$$

$s > 0$ $\qquad s > 6$

(b) $f(t) = \cos 2t - \sin 3t$

$$\mathscr{L}\{\cos 2t - \sin 3t\} = \mathscr{L}\{\cos 2t\} - \mathscr{L}\{\sin 3t\}$$

$$= \frac{s}{s^2+2^2} - \frac{3}{s^2+3^2} = \frac{s}{s^2+4} - \frac{3}{s^2+9}, \quad s>0$$

$s>0$ $\qquad s>0$

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$$\mathcal{L}\{(2-t)^2\} = \mathcal{L}\{4 - 4t + t^2\} = 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1}{s^2}\right) + \frac{2}{s^3} = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

for $s > 0$.

$s > 0$ $\quad$ $s > 0$ $\quad$ $s > 0$

(d) $f(t) = \sin^2 5t$ (In the interest of completeness)

$$= \frac{1}{2} - \frac{1}{2}\cos 10t$$

for $s > 0$

$$\mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos 10t\right\} = \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 10t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s}{s^2+10^2} = \frac{1}{2s} - \frac{s}{2(s^2+100)}$$