

Section 7.1: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s, t) = e^{-st}$.

Find the Laplace transform of $f(t) = 1$

By definition $\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} \cdot 1 \, dt$

If $s=0$, $e^{-st} = e^0 = 1$, $\int_0^{\infty} 1 \, dt$ diverges
zero is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$,

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^{\infty} e^{-st} \, dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} \, dt \\ &= \lim_{b \rightarrow \infty} \left. \frac{1}{-s} e^{-st} \right|_0^b\end{aligned}$$

$$= \lim_{b \rightarrow \infty} \frac{-1}{s} (e^{-sb} - e^0)$$

Divergent if
 $s < 0$

$$\text{for } s > 0 = \frac{-1}{s} (0 - 1) = \frac{1}{s}$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ with domain $s > 0$.

Find the Laplace transform of $f(t) = t$

By definition $\mathcal{L}\{t\} = \int_0^{\infty} e^{-st} t \, dt$

If $s=0$, the integral is $\int_0^{\infty} t \, dt$ which diverges

For $s \neq 0$, $\mathcal{L}\{t\} = \int_0^{\infty} e^{-st} t \, dt$

Integrate by Parts

$$u = t \quad du = dt$$

$$v = -\frac{1}{s} e^{-st} \quad dv = e^{-st} dt$$

$$= -\frac{1}{s} t e^{-st} \Big|_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt$$

$$\text{for } s > 0 \quad = -\frac{1}{s} (0 - 0 \cdot e^0) + \frac{1}{s} \underbrace{\int_0^{\infty} e^{-st} dt}_{\mathcal{L}\{1\}}$$

$$= \frac{1}{s} \left(\frac{1}{s} \right) = \frac{1}{s^2}$$

$$\text{Hence } \mathcal{L}\{t\} = \frac{1}{s^2} \quad \text{for } s > 0$$

Find the Laplace transform of $f(t) = e^{at}$, $a \neq 0$

By definition

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} \cdot e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt$$

If $s=a$, the integral $\int_0^{\infty} dt$ is divergent.

For $s \neq a$

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \frac{1}{-(s-a)} e^{-(s-a)t} \Big|_0^{\infty}$$

$$\text{for } s-a > 0 \quad = \frac{-1}{s-a} (0 - e^0)$$

$$= \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \text{ for } s > a$$

Find the Laplace transform¹ of $f(t) = \sin t$

$$\mathcal{L}\{\sin t\} = \int_0^{\infty} e^{-st} \sin t \, dt \qquad \int_0^{\infty} \sin t \, dt \text{ diverges}$$

$$\text{For } s \neq 0 \quad \mathcal{L}\{\sin t\} = \int_0^{\infty} e^{-st} \sin t \, dt$$

$$= \frac{e^{-st}}{(-s)^2 + 1^2} (-s \sin t - \cos t) \Big|_0^{\infty}$$

¹The following is immensely useful!

$$\int e^{\alpha t} \sin(\beta t) \, dt = \frac{e^{\alpha t}}{\alpha^2 + \beta^2} (\alpha \sin(\beta t) - \beta \cos(\beta t)).$$

$$\text{For } s > 0 \quad = 0 - \frac{e^0}{s^2 + 1} (-s \sin 0 - \cos 0)$$

$$= \frac{-1}{s^2 + 1} (0 - 1)$$

$$= \frac{1}{s^2 + 1}$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1} \quad \text{for } s > 0$$