October 22 Math 2306 sec. 53 Fall 2018

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_\alpha^\beta G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

Note 2: If we take s to be real-valued, then

$$\lim_{t\to\infty} e^{-st} = 0 \quad \text{if } s>0, \quad \text{and} \quad \lim_{t\to\infty} e^{-st} = \infty \quad \text{if } s<0.$$

Find the Laplace transform of $f(t) = 1$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^\infty e^{-st}\cdot 1\, dt$$

We have to consider $s=0$ separate from $s \neq 0$.

Case 1: $s=0$, then $e^{-0t} = 1$

$$\int_0^\infty 1\, dt = \lim_{b\to\infty}\int_0^b dt = \lim_{b\to\infty} t\Big|_0^b = \lim_{b\to\infty}(b-0) = \infty$$

The integral is divergent. Zero is not in the domain of $\mathcal{L}\{1\}$.

Case 2: $s \neq 0$ $\quad \mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \lim_{b\to\infty}\int_0^b e^{-st}\, dt$

$$= \lim_{b\to\infty} \frac{-1}{s} e^{-st}\Big|_0^b = \lim_{b\to\infty} \frac{-1}{s} e^{-sb} - \left(\frac{-1}{s} e^{0}\right)$$

diverges if $s<0$

$$= \frac{-1}{s}\cdot 0 + \frac{1}{s} \quad \text{for } s>0$$

$$= \frac{1}{s} \quad \text{for } s>0$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ with domain $s>0$.

Find the Laplace transform of $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

Case $\underline{1}$: $s = 0$ the integral is $\int_0^\infty t\,dt = \frac{t^2}{2}\Big|_0^\infty = \infty$

The integral diverges

Case 2: $s \neq 0$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\, dt$

$$= \frac{-t}{s} e^{-st}\Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

This is ∞ if $s<0$

$\mathcal{L}\{1\}$

* If $s>0$

$\lim_{t\to\infty} t e^{-st} = 0$

$$= 0 - 0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt \quad \text{for } s>0.$$

$$= \frac{1}{s}\mathcal{L}\{1\}$$

$$= \frac{1}{s}\left(\frac{1}{s}\right) \quad \text{for } s>0$$

$$= \frac{1}{s^2} \quad \text{for } s>0$$

$\Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$ with domain $s>0$.

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{10} e^{-st} f(t)\, dt + \int_{10}^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{10} e^{-st}(2t)\, dt + \int_{10}^\infty e^{-st} \cdot 0\, dt$$

$$= \int_0^{10} 2t\, e^{-st}\, dt$$

When $s=0$, the integral is

$$\int_0^{10} 2t\,dt = t^2 \Big|_0^{10} = 100$$

For $s \neq 0$

$$\mathcal{L}\{f(t)\} = \int_0^{10} 2t\,e^{-st}\,dt$$

$$= \frac{-2t}{s} e^{-st} \Big|_0^{10} - \frac{2}{s^2} e^{-st} \Big|_0^{10}$$

$$= \frac{-20}{s} e^{-10s} - 0 - \left(\frac{2}{s^2} e^{-10s} - \frac{2}{s^2} e^{0} \right)$$

$$= \frac{-20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s} + \frac{2}{s^2}$$

$$\mathcal{L}\{f(t)\} = \begin{cases} 100 & , \quad s = 0 \\ \dfrac{2}{s^2} - \dfrac{2}{s^2}e^{-10s} - \dfrac{20}{s}e^{-10s} & , \quad s \neq 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate the Laplace transform of

(a) $f(t) = \cos(\pi t)$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}$$

Examples: Evaluate

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2t^4 - e^{-5t} + 3\}$$

$$= 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\left(\frac{4!}{s^{4+1}}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right)$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}$$

valid for $s > 0$