October 23 Math 2306 sec 51 Fall 2015

Section 7.1: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \dots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Use the Table to Evaluate $\mathscr{L}\{f(t)\}$

Recall

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

(d) $f(t) = \sin^2 5t$

$$f(t) = \frac{1}{2} - \frac{1}{2}\cos(10t)$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(10t)\right\}$$

$$= \frac{1}{2}\mathscr{L}\{1\} - \frac{1}{2}\mathscr{L}\{\cos(10t)\}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+10^2} = \frac{1}{2s} - \frac{s}{2(s^2+100)}, \quad s>0$$

Hyperbolic Sine and Cosine

Define the hyperbolic sine and cosine functions

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \text{and} \quad \cosh(t) = \frac{e^t + e^{-t}}{2}, \quad \text{respectively.}$$

Find the Laplace transforms of sinh t and cosh t.

$$\mathcal{L}\{\sinh(t)\} = \mathcal{L}\left\{\frac{1}{2}e^t - \frac{1}{2}e^{-t}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^t\} - \frac{1}{2}\mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{2}\,\frac{1}{s-1} - \frac{1}{2}\,\frac{1}{s-(-1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

for $s > 1$

$s > 1$ $\qquad$ $s > -1$

$$\mathcal{L}\{\sinh t\} = \frac{1}{2}\left(\frac{s+1-(s-1)}{(s+1)(s-1)}\right) = \frac{1}{2}\,\frac{2}{s^2-1}$$

$$\mathcal{L}\{\cosh t\} = \mathcal{L}\left\{\frac{1}{2}e^{t} + \frac{1}{2}e^{-t}\right\}$$

$$= \frac{1}{2}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s+1} \qquad s > 1$$

$$= \frac{1}{2}\left(\frac{s+1+s-1}{(s+1)(s-1)}\right) = \frac{1}{2}\,\frac{2s}{s^2-1}$$

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}, \quad s>1$$

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2-1}, \quad s>1$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

some functions don't have Laplace transforms

$$f(t) = \frac{1}{t}.$$

Section 7.2: Inverse Transforms and Derivatives

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \Longleftrightarrow \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ the inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

From the table

$$\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = t^6$$

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\}$

Note $\dfrac{1}{s^7} = \dfrac{1}{6!}\,\dfrac{6!}{s^7}$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\}$$

$$= \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}t^6$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \cos 3t + \frac{1}{3}\sin 3t$$