Oct. 24 Math 1190 sec. 51 Fall 2016

Section 4.3: The Mean Value Theorem

Another significant consequence of the MVT is that it provides a test for the increasing and decreasing behavior of a differentiable function.

Theorem: Let f be differentiable on an open interval (a, b). If

- $f'(x) > 0$ on (a, b), the f is increasing on (a, b), and
- $f'(x) < 0$ on (a, b), the f is decreasing on (a, b).

Example

Determine the intervals over which f is increasing and the intervals over which it is decreasing where

$$f(x) = 2x^3 - 6x^2 - 18x + 1$$

We did this problem on Friday. The domain of f is all reals. We found that $f'(x) = 6(x-3)(x+1)$ so that $f'(x) = 0$ when $x = 3$ and when $x = -1$. We

- split the real line up by these numbers,
- tested the sign in each interval by putting a test value into $f'(x)$, and
- recorded the signs.

Based on that, we determined that f is increasing on $(-\infty, -1) \bigcup (3, \infty)$ and decreasing on $(-1, 3)$.

Question

Suppose that we compute the derivative of some function g and find

$$g'(x) = (2 + x)e^{x/2}.$$

Determine the intervals over which g is increasing and over which it is decreasing.

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-2

(a) g is increasing on $(-1/2, \infty)$ and decreasing on $(-\infty, -1/2)$.

(b) g is increasing on $(-2, \infty)$ and decreasing on $(-\infty, -2)$.

(c) g is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

(d) g is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.

Section 4.4: Local Extrema and Concavity

We have already seen that the first derivative f' can tell us about the behaviour of the function f—in particular, it gives information about where it is increasing or decreasing, and where it may take a local extreme value.

In this section, we'll expand on that as well as introduce information about a function that can be deduced from the nature of its second derivative.

Theorem: First derivative test for local extrema

Let f be continuous and suppose that c is a critical number of f.

- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

Note: we read from left to right as usual when looking for a sign change.

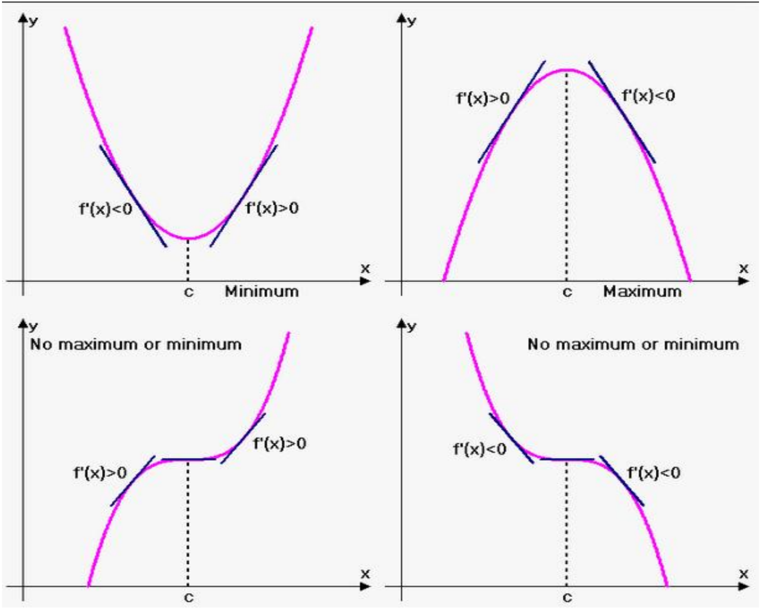


Figure: First derivative test

Example

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$f(x) = x^{1/3}(16 - x) = 16x^{1/3} - x^{4/3}$$

The domain is $(-\infty, \infty)$.

Find the critical points:

$$f'(x) = 16\left(\frac{1}{3}x^{-2/3}\right) - \frac{4}{3}x^{1/3}$$

$$= \frac{16}{3x^{2/3}} - \frac{4x^{1/3}}{3} \cdot \frac{x^{2/3}}{x^{2/3}}$$

$$= \frac{16}{3x^{2/3}} - \frac{4x}{3x^{2/3}} = \frac{16-4x}{3x^{2/3}}$$

$$f'(x) = \frac{16-4x}{3x^{2/3}}$$

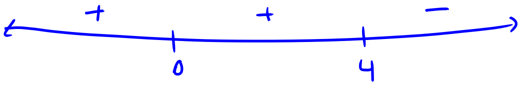
$f'(x) = 0$ if the numerator is zero

$$16-4x = 0 \Rightarrow 16 = 4x \Rightarrow x = 4$$

$f'(x)$ is undefined if the denominator is zero.

$$3x^{2/3} = 0 \Rightarrow x = 0$$

Do a sign analysis on $f'(x) = \dfrac{16-4x}{3x^{2/3}} = \dfrac{16-4x}{3\sqrt[3]{x^2}}$



Test -1 : $f'(-1) = \dfrac{16-4(-1)}{3\sqrt[3]{(-1)^2}} = \dfrac{20}{3}$

1 : $f'(1) = \dfrac{16-4\cdot 1}{3\sqrt[3]{1^2}} = \dfrac{12}{3}$

5 : $f'(5) = \dfrac{16-4\cdot 5}{3\sqrt[3]{(5)^2}} = \dfrac{-4}{3\sqrt[3]{25}}$

The sign of f' doesn't change @ $x=0$.

There is no local extremum there.

The sign of f' changes from $+$ to $-$

@ $x=4$. f has a local maximum

at $x=4$.

Question

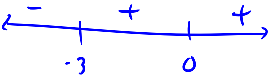
Consider the function $f(t) = t^4 + 4t^3$. Which of the following is true about this function?

$f'(t) = 4t^3 + 12t^2 = 4t^2(t+3)$

(a) f has a local minimum at $t = 0$ and a local maximum at $t = -3$.

(b) f has a local minimum at $t = -3$ and a local maximum at $t = 0$.

(c) f has a local minimum at $t = -3$.

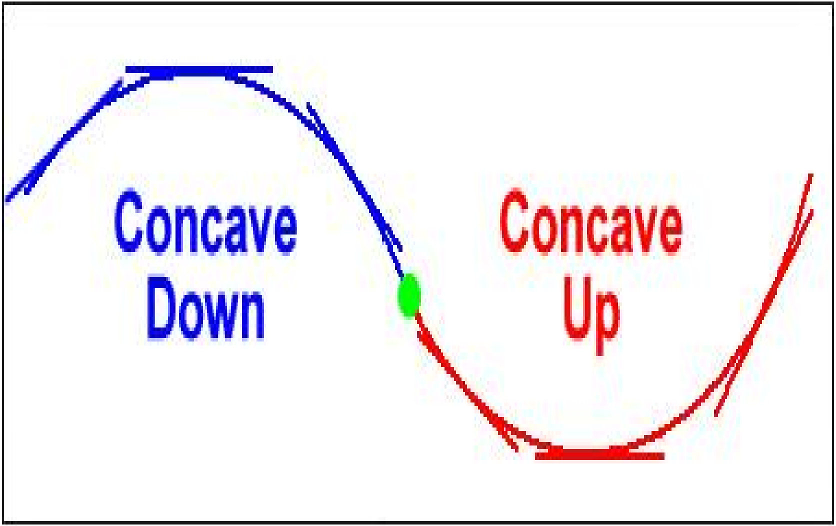


(d) f has a local minimum at $t = 0$.

Concavity and The Second Derivative

Concavity: refers to the *bending* nature of a graph. In particular, a curve is concave down if it's cupped side is down, and it is concave up if it's cupped upward.

Concavity



Figure

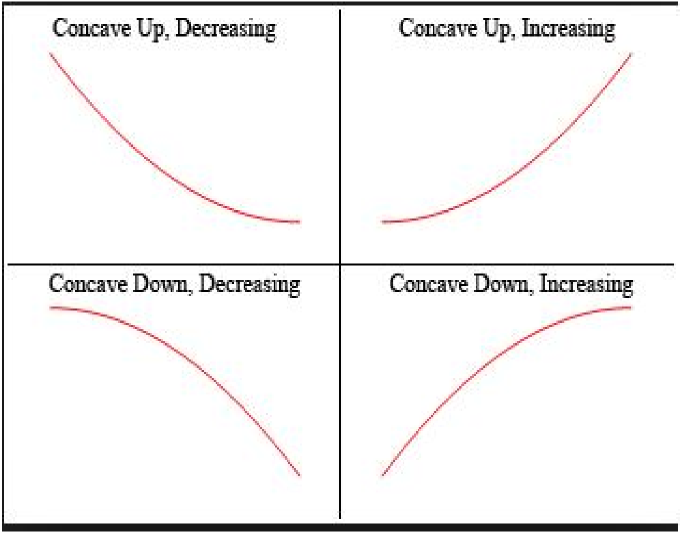


Figure: A graph can have either increasing or decreasing behavior and be either concave up or down.

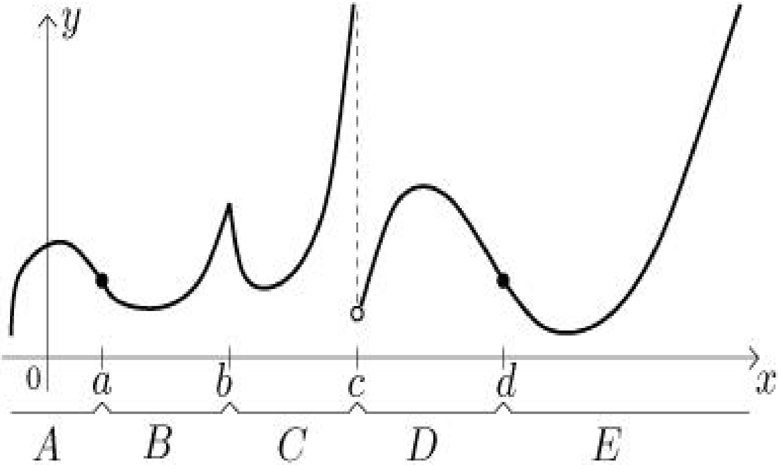


Figure: We can consider concavity at a point, but it's best thought of as a property over an interval. Many function's graphs have concavity that changes over the domain.

Definition of Concavity

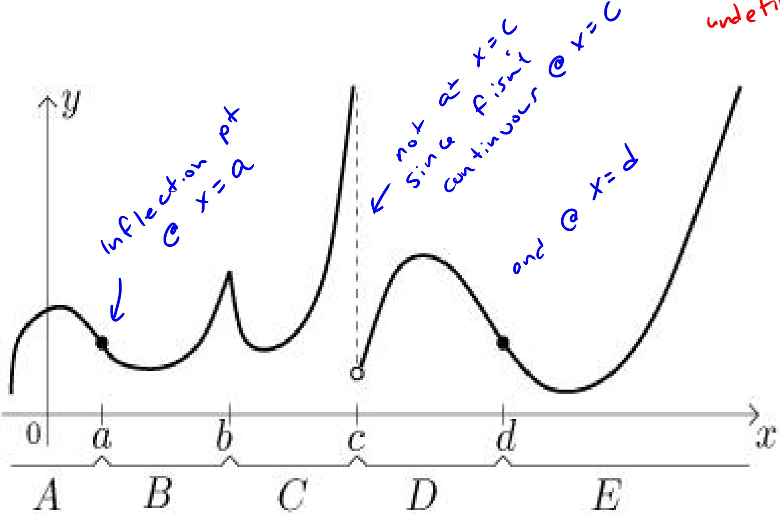
If the graph of a function f lies above all of its tangent lines over an interval I, then f is concave up on I. If the graph of f lies below each of its tangent lines on an interval I, f is concave down on I.

Theorem: (Second Derivative Test for Concavity)
Suppose f is twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of f is concave up on I.
- If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P (from down to up or from up to down). A point where $f''(x) = 0$ would be a candidate for being an inflection point.

or where $f''(x)$ is undefined.



Concavity and Extrema:

Theorem: (Second Derivative Test for Local Extrema)
Suppose $f'(c) = 0$ and that f'' is continuous near c. Then

- if $f''(c) > 0$, f takes a local minimum at c,
- if $f''(c) < 0$, then f takes a local maximum at c.

If $f''(c) = 0$, then the test fails. f may or may not have a local extrema. You can go back to the first derivative test to find out.

Example

Analyze the function $f(x) = xe^{3x}$. In particular, indicate

(a) the intervals on which f is increasing and decreasing, ← 1st der.

(b) the intervals on which f is concave up and concave down, ← 2nd der.

(c) identify critical points and classify any local extrema, and

(d) identify any points of inflection. ← 2nd der

Find f' and f''

$$f'(x) = 1 \cdot e^{3x} + x\, e^{3x} \cdot 3 = e^{3x}(1+3x)$$

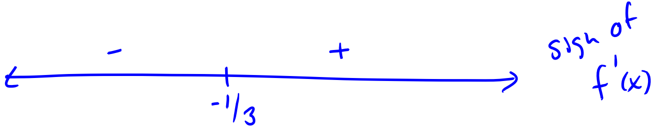
$$f''(x) = e^{3x} \cdot 3 + 1 \cdot e^{3x} \cdot 3 + x\, e^{3x} \cdot 3 \cdot 3 = e^{3x}(6+9x)$$

Sign analysis on $f'(x)$: The domain of f is $(-\infty, \infty)$

$$f'(x) = 0 \Rightarrow e^{3x}(1+3x) = 0$$

$$\Rightarrow e^{3x} = 0 \text{ (no solutions) or } 1+3x = 0 \Rightarrow x = \frac{-1}{3}$$

$f'(x)$ is undefined never.



Test $f'(-1) = e^{-3}(1-3) = -2e^{-3}$

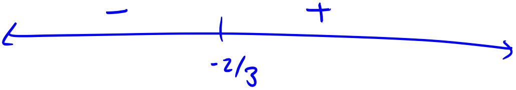
$f'(0) = e^{0}(1+0) = e^{0} = 1$

(a) f is decreasing on $(-\infty, -\frac{1}{3})$ and f is increasing on $(-\frac{1}{3}, \infty)$

Do a sign analysis on $f''(x) = e^{3x}(6+9x)$

$f''(x) = 0 \Rightarrow e^{3x}(6+9x) = 0$ $\quad e^{3x} \neq 0$ for all x

$6+9x = 0 \Rightarrow 9x = -6 \Rightarrow x = \frac{-6}{9} = \frac{-2}{3}$



Sign of $f''(x)$

Test $f''(-1) = e^{-3}(6-9) = -3e^{-3}$

$f''(0) = e^{0}(6+0) = 6e^{0} = 6$

(b) f is concave down on $(-\infty, \frac{-2}{3})$ and f is concave up on $(\frac{-2}{3}, \infty)$

(d) f has an inflection pt. @ $x = \frac{-2}{3}$ since the concavity changes there.

(c) f has one critical number $\frac{-1}{3}$. The graph takes a local minimum there by the 1st (or 2nd) derivative test.

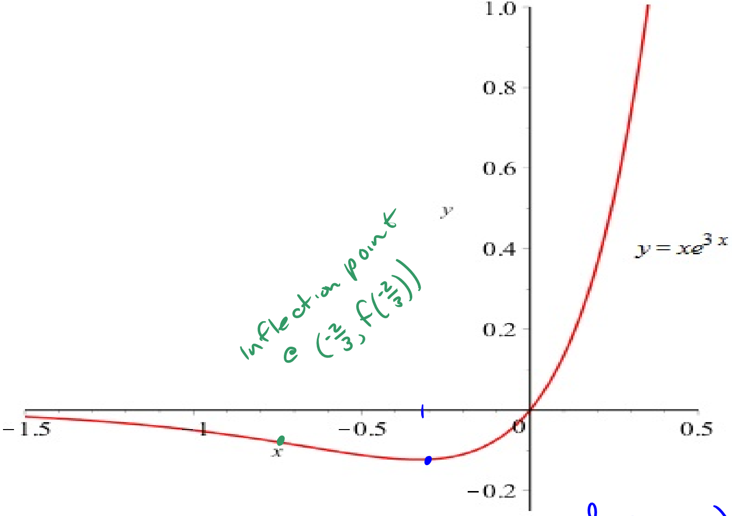


Figure: Plot of $y = xe^{3x}$.

local min @ $\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right)$

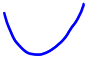
Questions

(1) **True or False** If $f''(2) = 0$ it must be that f has an inflection point $(2, f(2))$.

False, concavity need not change. E.g. $f(x) = (x-2)^4$

(2) Suppose that we know a function f satisfies the two conditions $f'(1) = 0$ and $f''(1) = 4$. Which of the following can we conclude with certainty?

(a) f has a local minimum at $(1, f(1))$.



2nd der. test tells us we have a local min.

(b) f has an inflection point at $(1, f(1))$.

(c) f has a local maximum at $(1, f(1))$.

(d) None of the above are necessarily true.