Oct. 24 Math 1190 sec. 52 Fall 2016

Section 4.4: Local Extrema and Concavity

Theorem: First derivative test for local extrema
Let f be continuous and suppose that c is a critical number of f.

- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

Note: we read from left to right as usual when looking for a sign change.

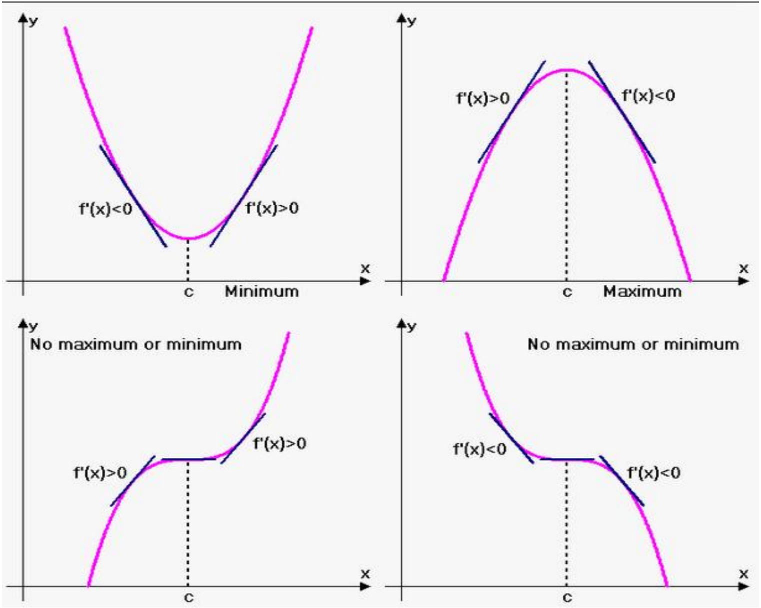


Figure: First derivative test

Example

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$f(x) = x^{1/3}(16 - x)$$

$$f'(x) = \frac{16-4x}{3x^{2/3}}$$

We did this problem on Friday. The domain of f is all reals. We found the critical points to be 4 and 0. We split the domain up by these numbers, and found that

$$f'(x) > 0 \quad \text{if} \quad -\infty < x < 0 \quad \text{or} \quad 0 < x < 4$$

$$\text{and} \quad f'(x) < 0 \quad \text{if} \quad x > 4.$$

We determined that the critical number 0 doesn't correspond to a local extremum, and 4 is where f takes a local max.

Question

Consider the function $f(t) = t^4 + 4t^3$. Which of the following is true about this function?

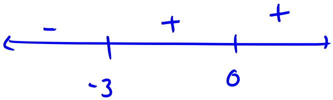
$f'(t) = 4t^3 + 12t^2 = 4t^2(t+3)$ crit # 0, -3

(a) f has a local minimum at $t = 0$ and a local maximum at $t = -3$.

(b) f has a local minimum at $t = -3$ and a local maximum at $t = 0$.

(c) f has a local minimum at $t = -3$.

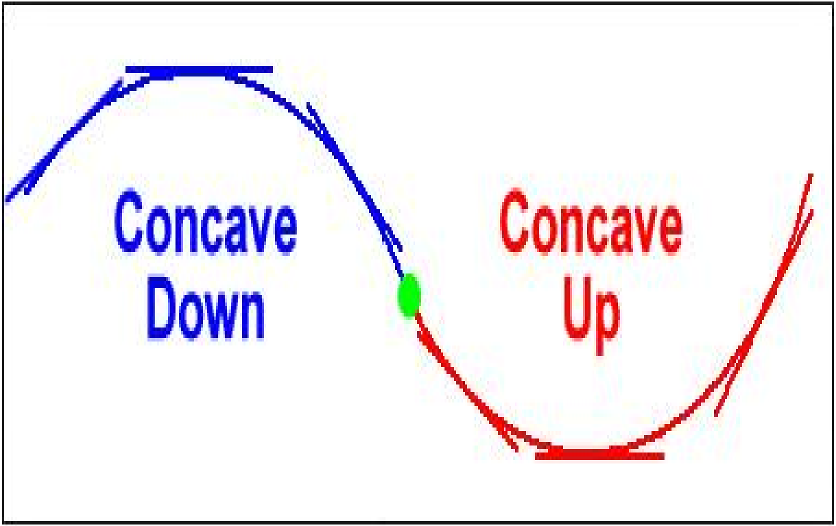
(d) f has a local minimum at $t = 0$.



Concavity and The Second Derivative

Concavity: refers to the *bending* nature of a graph. In particular, a curve is concave down if it's cupped side is down, and it is concave up if it's cupped upward.

Concavity



Figure

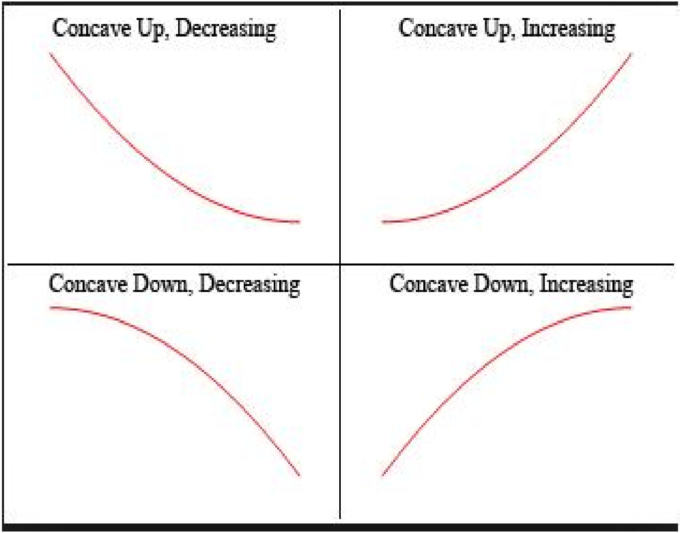


Figure: A graph can have either increasing or decreasing behavior and be either concave up or down.

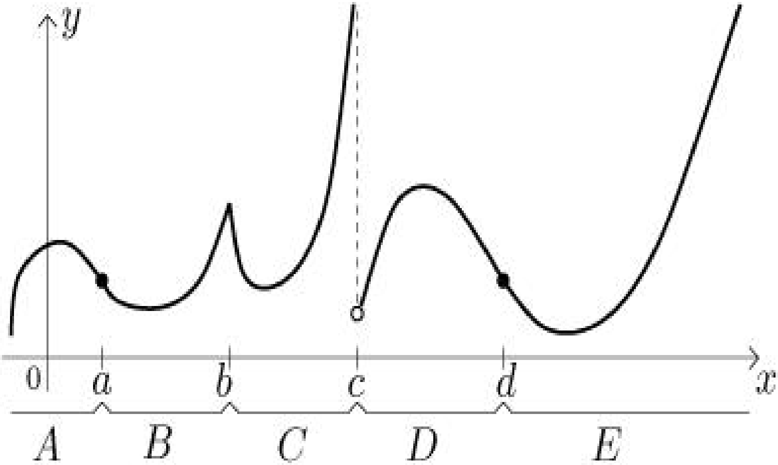


Figure: We can consider concavity at a point, but it's best thought of as a property over an interval. Many function's graphs have concavity that changes over the domain.

Definition of Concavity

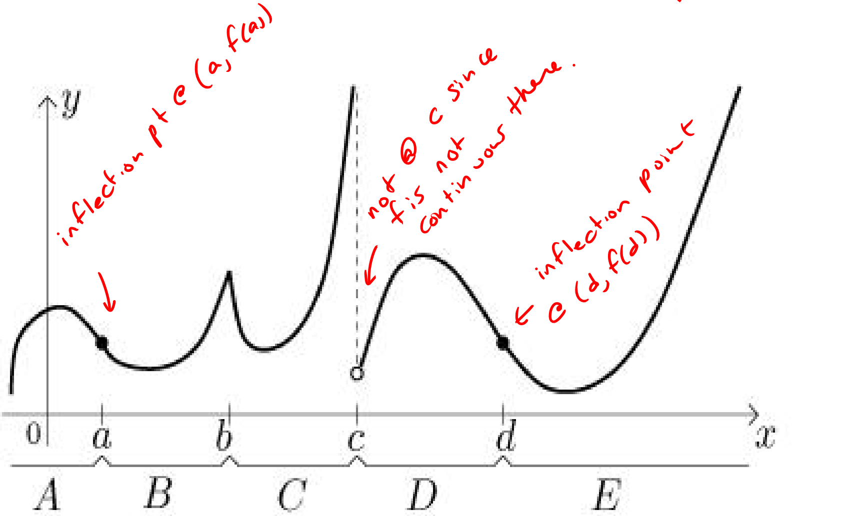
If the graph of a function f lies above all of its tangent lines over an interval I, then f is concave up on I. If the graph of f lies below each of its tangent lines on an interval I, f is concave down on I.

Theorem: (Second Derivative Test for Concavity)
Suppose f is twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of f is concave up on I.
- If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P (from down to up or from up to down). A point where $f''(x) = 0$ would be a candidate for being an inflection point.

or where $f''(x)$ is undefined



Concavity and Extrema:

Theorem: (Second Derivative Test for Local Extrema)
Suppose $f'(c) = 0$ and that f'' is continuous near c. Then

- if $f''(c) > 0$, f takes a local minimum at c,
- if $f''(c) < 0$, then f takes a local maximum at c.

If $f''(c) = 0$, then the test fails. f may or may not have a local extrema. You can go back to the first derivative test to find out.

Example

Analyze the function $f(x) = xe^{3x}$. In particular, indicate

(a) the intervals on which f is increasing and decreasing, 1st der.

(b) the intervals on which f is concave up and concave down, 2nd der.

(c) identify critical points and classify any local extrema, and Both

(d) identify any points of inflection. 2nd der.

$$f'(x) = 1 \cdot e^{3x} + x e^{3x} \cdot 3 = e^{3x}(1+3x)$$

$$f''(x) = e^{3x} \cdot 3 + 1 \cdot e^{3x} \cdot 3 + x \cdot 3 e^{3x} \cdot 3 = e^{3x}(6+9x)$$

Sign analysis on $f'(x) = e^{3x}(1+3x)$

$f'(x)=0 \quad \Rightarrow \quad e^{3x}(1+3x) = 0$

$\Rightarrow e^{3x} = 0$ (no solutions since $e^{3x} > 0$ for all x)

or $1+3x = 0 \quad \Rightarrow \quad x = \frac{-1}{3}$

$f'(x)$ is never undefined.

The domain of f is $(-\infty,\infty)$. There is one critical number $x = \frac{-1}{3}$.

$$f'(x) = e^{3x}(1+3x)$$

Sign of f': $-$ on the left of $-\frac{1}{3}$, $+$ on the right of $-\frac{1}{3}$

Test:
$$f'(-1) = e^{-3}(1-3) = -2e^{-3}$$

$$f'(0) = e^{0}(1+0) = e^{0}\cdot 1 = 1$$

(a) The graph of f is decreasing on $(-\infty, -\frac{1}{3})$, and it's increasing on $(-\frac{1}{3}, \infty)$.

(c) f has a local minimum @ $-\frac{1}{3}$, the only critical point. It's a local min by the

1st derivative test.

Do a sign analysis on $f''(x) = e^{3x}(6+9x)$

$f''(x) = 0 \Rightarrow e^{3x}(6+9x) = 0$ $e^{3x} > 0$ for all x

$6+9x=0 \Rightarrow 9x=-6 \Rightarrow x = \frac{-6}{9} = \frac{-2}{3}$

$f''(x)$ is never undefined.

$f''(x) = e^{3x}(6+9x)$

Sign of $f''(x)$

$$\begin{array}{ccc} - & & + \\ \hline & -\frac{2}{3} & \end{array}$$

Test: $f''(-1) = e^{-3}(6-9) = -3e^{-3}$

$f''(0) = e^{0}(6+0) = 6\cdot 1 = 6$

(b) The graph of f is concave down on $\left(-\infty, \frac{-2}{3}\right)$ and concave up on $(-2/3, \infty)$.

(d) Concavity changes @ $\frac{-2}{3}$, so there is an inflection point @ $\left(\frac{-2}{3}, f\left(\frac{-2}{3}\right)\right)$, by the sign of $f''(x)$.

← 2^{nd} derivative test for local extrema

Note $f''\left(\frac{-1}{3}\right) = e^{-1}(6-3) = 3e^{-1} > 0$

confirming that f has a local minimum @ $\left(\frac{-1}{3}, f\left(\frac{-1}{3}\right)\right)$.

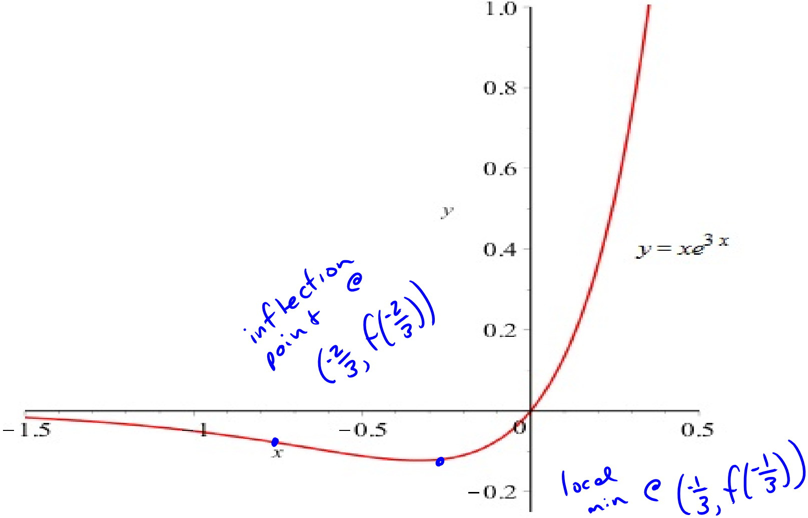


Figure: Plot of $y = xe^{3x}$.

Questions

(1) **True or False** If $f''(2) = 0$ it must be that f has an inflection point $(2, f(2))$.

False, concavity need not change e.g. $f(x) = (x-2)^4$

(2) Suppose that we know a function f satisfies the two conditions $f'(1) = 0$ and $f''(1) = 4$. Which of the following can we conclude with certainty?

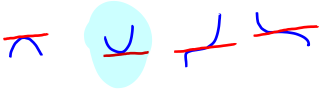
$f'(1) = 0 \to$ horizontal tangent

(a) f has a local minimum at $(1, f(1))$.

(b) f has an inflection point at $(1, f(1))$.

$f''(1) = 4 > \to$ concave up

(c) f has a local maximum at $(1, f(1))$.



(d) None of the above are necessarily true.

Section 4.5: Indeterminate Forms & L'Hôpital's Rule

Consider the following three limit statements (all of which are true):

(a) $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}=2$

"$\frac{0}{0}$" isn't defined.

(b) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$

(c) $\displaystyle\lim_{x\to 3}\frac{x^2-9}{(x-3)^2}$ doesn't exist

Note: Each of these three limits involve both numerator and denominator going to zero—giving the form $\frac{0}{0}$. In the top two, the limit exists, but the limits are different. In the third, the limit doesn't exist.

Indeterminate Forms

$0/0$ is called an **Indeterminate form**.

Other indeterminate forms we'll encounter include

$$\frac{\pm\infty}{\pm\infty}, \quad \infty - \infty, \quad 0\infty, \quad 1^{\infty}, \quad 0^{0}, \quad \text{and} \quad \infty^{0}.$$

Indeterminate forms are not defined (as number)

Theorem: l'Hospital's Rule

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0$$

← indeterminate form $\frac{0}{0}$

OR if

$$\lim_{x \to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \pm\infty$$

← or $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$, $\frac{-\infty}{-\infty}$, $\frac{\infty}{-\infty}$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Evaluate each limit if possible

(a) $\lim_{x \to 1} \frac{\ln x}{x-1}$ $=$ "$\frac{0}{0}$" since $\lim_{x \to 1} \ln x = \ln 1 = 0$

and

$\lim_{x \to 1} x - 1 = 1 - 1 = 0$

use l'H rule

$$= \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \frac{\frac{1}{1}}{1} = 1$$