October 24 Math 2306 sec. 53 Fall 2018

Section 13: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate the Laplace transform of

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{4 - 4t + t^2\}$

$= 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$

$= 4 \cdot \frac{1}{s} - 4\frac{1!}{s^2} + \frac{2!}{s^3}$

$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$

Examples: Evaluate the Laplace transform of

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

(d) $f(t) = \sin^2 5t$

$= \frac{1}{2} - \frac{1}{2}\cos(10t)$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\frac{1}{2} - \frac{1}{2}\cos(10t)\}$

$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos(10t)\}$

$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\frac{s}{s^2+10^2} = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}s}{s^2+100}$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order c for some $c>0$, then f has a Laplace transform for $s>c$.

An example of a function that is NOT of exponential order for any c is $f(t)=e^{t^2}$. Note that

$$f(t)=e^{t^2}=\left(e^t\right)^t \quad\Longrightarrow\quad |f(t)|>e^{ct} \quad \text{whenever} \quad t>c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

From the table

$$\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = t^6 \quad (n=6 \text{ case})$$

Note that $\frac{1}{s^7} = \frac{1}{6!}\,\frac{6!}{s^7}$ so

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\} = \frac{1}{6!}\,\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}\,t^6$$

Example: Evaluate

$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

Note $\dfrac{s+1}{s^2+9} = \dfrac{s}{s^2+3^2} + \dfrac{1}{s^2+3^2} = \dfrac{s}{s^2+3^2} + \dfrac{1}{3}\,\dfrac{3}{s^2+3^2}$

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\,\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$