October 24 Math 2306 sec. 57 Fall 2017

We defined The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Section 14: Inverse Laplace Transforms

Given $F(s)$, $f(t)$ an inverse Laplace transform of $F(s)$,

$$\mathscr{L}^{-1}\{F(s)\} = f(t), \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

In general, we'll use one table to find Laplace transforms and inverse transforms.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

We'll do a partial fraction decomp on $\dfrac{s-8}{s^2-2s}$.

$$\frac{s-8}{s^2-2s} = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Clear fractions

$$s-8 = A(s-2) + Bs$$

$$s-8 = (A+B)s - 2A$$

Matching coefficients

$$A+B = 1$$

$$-2A = -8 \Rightarrow A = 4$$

$$B = 1-A = 1-4 = -3$$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4\cdot 1 - 3e^{2t}$$

$$= 4 - 3e^{2t}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

By definition $\quad \mathscr{L}\left\{e^t t^2\right\}=\displaystyle\int_0^\infty e^{-st}e^t t^2\,dt$

Note:

$e^{-st}\cdot e^{t} = e^{-st+t}$

$= e^{-(s-1)t}$

$= \int_0^\infty e^{-(s-1)t}\, t^2\, dt$

$= \int_0^\infty e^{-wt}\, t^2\, dt$ if $w=s-1$

If $F(s)=\mathscr{L}\{t^2\}$ this is $F(w)$

Observe that this is simply the Laplace transform of $f(t)=t^2$ evaluated at $s-1$. Letting $F(s)=\mathscr{L}\left\{t^2\right\}$, we have

$$F(s-1)=\frac{2}{(s-1)^3}.$$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible

we'll complete the square.

$$s^2+2s+2 = s^2+2s+1+1 = (s+1)^2+1 = (s+1)^2+1^2$$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}$$

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

we'll use

$$s = s+1-1$$

we'd need every s replaced with $s+1$

$$\frac{s}{s^2+2s+2} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

We'll do a partial fraction decomp.

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions $s(s-1)^2$

$$1+3s-s^2 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2 + 3s + 1 = (A+B)s^2 + (-2A-B+C)s + A$$

$$-1 = A + B$$

$$3 = -2A - B + C$$

$$1 = A$$

$$B = -1 - A = -2, \quad C = 3 + 2A + B = 3 + 2(1) - 2 = 3$$

$$\mathcal{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s-1)^2}}_{\text{looks like } \frac{1}{s^2}}\right\}$$

$$= 1 - 2e^t + 3e^t t$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

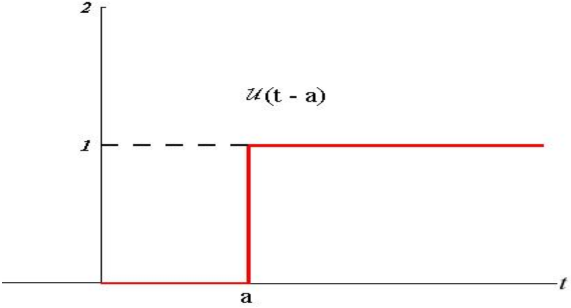

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

If $0 \le t < a$, $\mathscr{U}(t-a) = 0$, the right side is

$g(t) - g(t)\cdot 0 + h(t)\cdot 0 = g(t)$

If $t \ge a$, $\mathscr{U}(t-a) = 1$, the right side is

$g(t) - g(t)\cdot 1 + h(t)\cdot 1 = h(t)$

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

$f(t) = e^t - e^t\,\mathscr{U}(t-2) + t^2\,\mathscr{U}(t-2) - t^2\,\mathscr{U}(t-5) + 2t\,\mathscr{U}(t-5)$

Verify: for $0 \le t < 2$, $\mathscr{U}(t-2) = 0$ $\qquad \mathscr{U}(t-5) = 0$

$t < 2 \Rightarrow t < 5$

$f(t) = e^t - e^t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + 2t \cdot 0 = e^t$

For $2 \le t < 5$, $\mathcal{U}(t-2) = 1$ $\mathcal{U}(t-5) = 0$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + 2t \cdot 0 = t^2$$

For $t \geq 5$, $\mathcal{U}(t-2) = 1$ $\mathcal{U}(t-5) = 1$

$t \geq 5 \Rightarrow t \geq 2$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + 2t \cdot 1 = 2t$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

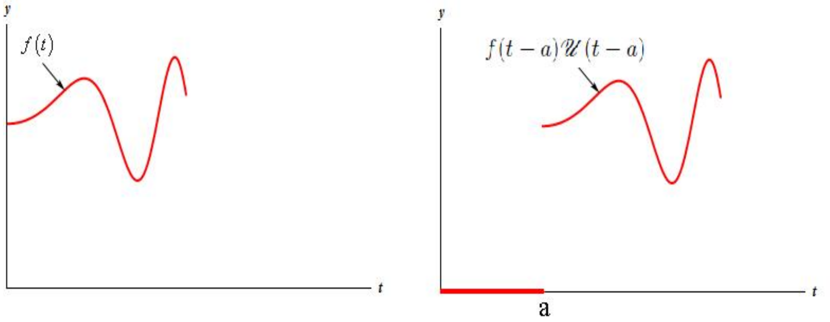

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$