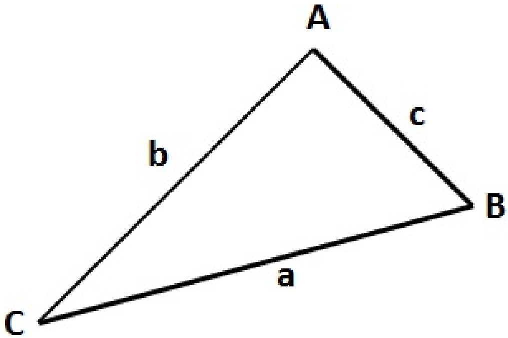
October 24 Math 3260 sec. 57 Fall 2017

Recall the Law of Cosines

For triangle with angles A, B, C and opposite sides of lengths a, b, and c, respectively,

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$



Geometry of the Dot Product

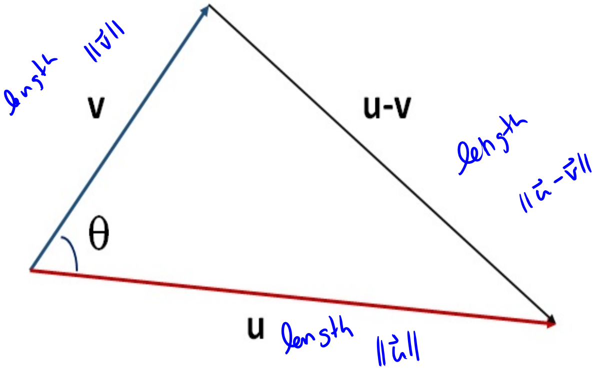


Figure: We can use the law of cosines to show that in $\mathbb{R}^2$ that $\mathbf{u}\cdot\mathbf{v}$ is related to the angle between the two (nonzero) vectors. This holds in $\mathbb{R}^n$. We're just restricting n to 2 for ease of computation.

By the law of Cosines

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

We can note that $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$

$$\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

$$\cancel{\|\vec{u}\|^2} - 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{v}\|^2} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

$$-\cancel{2}\vec{u}\cdot\vec{v} = -\cancel{2}\|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

Section 6.2: Orthogonal Sets

Remark: We know that if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace W of $\mathbb{R}^n$, then each vector $\mathbf{x}$ in W can be realized (uniquely) as a sum

$$\mathbf{x} = c_1\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p.$$

If n is very large, the computations needed to determine the coefficients $c_1, \ldots, c_p$ may require a lot of time (and machine memory).

Question: Can we seek a basis whose nature simplifies this task? And what properties should such a basis possess?

Orthogonal Sets & Orthogonal Bases

Definition: An indexed set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** provided each pair of distinct vectors in the set is orthogonal. That is, provided

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{whenever} \quad i \neq j.$$

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis that is also an orthogonal set.

Theorem: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then each vector $\mathbf{y}$ in W can be written as the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p, \quad \text{where the weights}$$

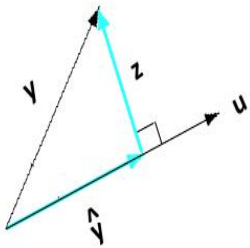
$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}.$$

Projection

Given a nonzero vector **u**, suppose we wish to decompose another nonzero vector **y** into a sum of the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

in such a way that $\hat{\mathbf{y}}$ is parallel to **u** and **z** is perpendicular to **u**.



$\vec{z} = \vec{y} - \hat{y}$

Projection

Since $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$, there is a scalar α such that

$$\hat{\mathbf{y}} \;=\; \alpha\mathbf{u}.$$

We determined last time that the scalar

$$\alpha = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}$$

so that

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

Projection onto the subspace $L = \text{Span}\{\mathbf{u}\}$

Notation: $\hat{\mathbf{y}} = \text{proj}_L = \left(\dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$

Example: Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is in Span$\{\mathbf{u}\}$ and $\mathbf{z}$ is orthogonal to $\mathbf{u}$.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \left(\frac{7(4)+6(2)}{4^2+2^2}\right)\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So $\vec{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example Continued...

Determine the distance between the point $(7, 6)$ and the line Span$\{\mathbf{u}\}$.

The shortest distance is $\|\vec{z}\|$.

The distance is $\sqrt{(-1)^2 + 2^2} = \sqrt{5}$

Orthonormal Sets

Definition: A set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called an **orthonormal set** if it is an orthogonal set of **unit vectors.**

Definition: An **orthonormal basis** of a subspace W of $\mathbb{R}^n$ is a basis that is also an orthonormal set.

Example: Show that $\left\{ \underset{\vec{u}_1}{\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

$\vec{u}_1 \cdot \vec{u}_2 = \frac{3}{5}\left(\frac{-4}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{-16+16}{25} = 0$ They are orthogonal

$\|\vec{u}_1\|^2 = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{16+9}{25} = \frac{25}{25} = 1$, $\|\vec{u}_2\|^2 = \left(\frac{-4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16+9}{25} = 1$

They are also unit vectors.

If the vectors are linearly independent, they span $\mathbb{R}^2$ (since there are 2 of them).

$$c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$$

$$\vec{u}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix},$$

$\vec{u}_1$ is not a scaled version on $\vec{u}_2$.

Hence they are lin. independent and $\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

Orthogonal Matrix

Consider the matrix $U = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ whose columns are the vectors in the last example. Compute the product

$$U^T U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that $U^T = U^{-1}$

What does this say about U^{-1}?

Orthogonal Matrix

Definition: A square matrix U is called an **orthogonal matrix** if $U^T = U^{-1}$.

Theorem: An $n \times n$ matrix U is orthogonal if and only if it's columns form an orthonormal basis of $\mathbb{R}^n$.

The linear transformation associated to an orthogonal matrix preserves *lenghts* and *angles* in the following sense:

Theorem: Orthogonal Matrices

Let U be an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ vectors in $\mathbb{R}^n$. Then

(a) $\|U\mathbf{x}\| = \|\mathbf{x}\|$

(b) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, in particular

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof (of (a)):

Recall $\|U\vec{x}\|^2 = (U\vec{x}) \cdot (U\vec{x})$

$= (U\vec{x})^T (U\vec{x})$

This is how we defined the dot product

$$= (\vec{x}^T U^T)(U\vec{x})$$

Recall
$$(AB)^T = B^T A^T$$

$$= x^T (U^T U)\vec{x}$$

$$= \vec{x}^T I \vec{x}$$

$$= \vec{x}^T \vec{x} = \|\vec{x}\|^2$$

So

$$\|U\vec{x}\| = \|\vec{x}\|$$

Section 6.3: Orthogonal Projections

Equating points with position vectors, we may wish to find the point $\hat{\mathbf{y}}$ in a subspace W of $\mathbb{R}^n$ that is *closest* to a given point $\mathbf{y}$.

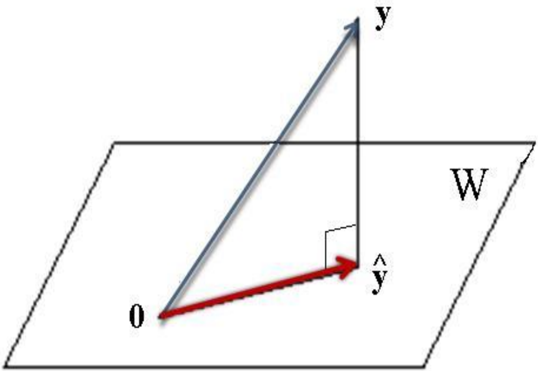


Figure: Illustration of an orthogonal projection. Note that $\mathrm{dist}(\mathbf{y}, \hat{\mathbf{y}})$ is the shortest distance between $\mathbf{y}$ and the points on W.

Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Each vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as a sum

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$.

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis for W, then

$$\hat{\mathbf{y}} = \sum_{j=1}^{p} \left(\frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \right) \mathbf{u}_j, \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

Remark: Note that the basis must be orthogonal, but otherwise the vector $\hat{\mathbf{y}}$ is **independent** of the particular basis used!

Remark: The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of y onto** W. We can denote it

$$\mathrm{proj}_W\, \mathbf{y}.$$

Example

Let $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$ and

call these

$\vec{u}_1$ $\vec{u}_2$

$$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(a) Verify that the spanning vectors for W given are an orthogonal basis for W.

$\vec{u}_1 \cdot \vec{u}_2 = 2(-2) + 1(2) + 2(1) = -4+2+2 = 0$

They are orthogonal. This implies that they are also linearly independent hence an orthogonal basis.

Example Continued...

$$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$$

(b) Find the orthogonal projection of **y** onto W.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\|\vec{u}_2\|^2}\vec{u}_2$$

$$\|\vec{u}_1\|^2 = 2^2+1^2+2^2 = 9 \implies \|\vec{u}_1\| = 3$$

$$\|\vec{u}_2\|^2 = (-2)^2+2^2+1^2 = 9 \implies \|\vec{u}_2\| = 3$$

$$\vec{y}\cdot\vec{u}_1 = 2\cdot 4 + 1\cdot 8 + 2\cdot 1 = 18, \quad \vec{y}\cdot\vec{u}_2 = -2(4) + 2(8) + 1 = 9$$

$$\hat{y} = \frac{18}{9}\vec{u}_1 + \frac{9}{9}\vec{u}_2 = 2\vec{u}_1 + \vec{u}_2$$

$$= 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 2+2 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

(c) Find the shortest distance between **y** and the subspace W.

Let $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$

The shortest distance is

$$\|\vec{z}\| = \sqrt{2^2 + 4^2 + (-4)^2} = \sqrt{4+16+16} = 6$$