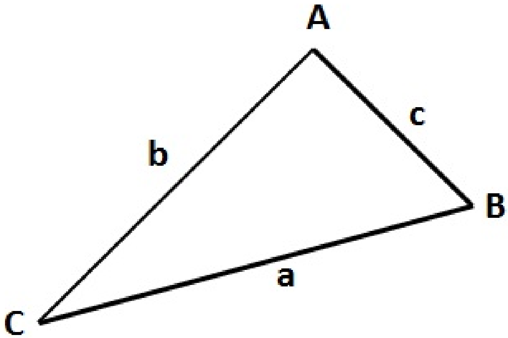
October 24 Math 3260 sec. 58 Fall 2017

Recall the Law of Cosines

For triangle with angles A, B, C and opposite sides of lengths a, b, and c, respectively,

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$



Geometry of the Dot Product

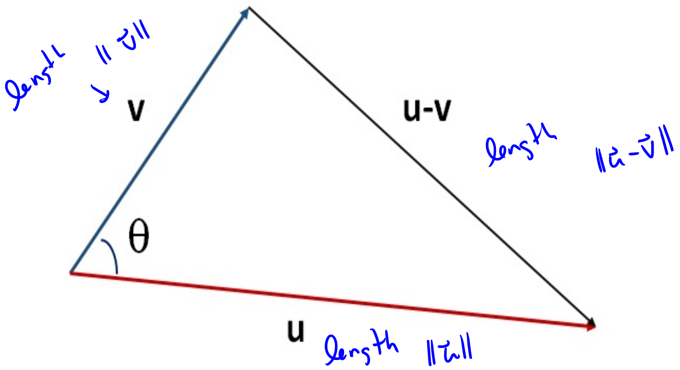


Figure: We can use the law of cosines to show that in $\mathbb{R}^2$ that $\mathbf{u} \cdot \mathbf{v}$ is related to the angle between the two (nonzero) vectors. This holds in $\mathbb{R}^n$. We're just restricting n to 2 for ease of computation.

By the law of cosines

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\begin{aligned}\|\vec{u}-\vec{v}\|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2\end{aligned}$$

$$\cancel{\|\vec{u}\|^2} - 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{v}\|^2} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$-\cancel{2}\,\vec{u}\cdot\vec{v} = -\cancel{2}\,\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\Rightarrow \qquad \vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

Section 6.2: Orthogonal Sets

Remark: We know that if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace W of $\mathbb{R}^n$, then each vector $\mathbf{x}$ in W can be realized (uniquely) as a sum

$$\mathbf{x} = c_1\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p.$$

If n is very large, the computations needed to determine the coefficients $c_1, \ldots, c_p$ may require a lot of time (and machine memory).

Question: Can we seek a basis whose nature simplifies this task? And what properties should such a basis possess?

Orthogonal Sets

Definition: An indexed set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** provided each pair of distinct vectors in the set is orthogonal. That is, provided

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{whenever} \quad i \neq j.$$

Example: Show that the set $\left\{ \underset{\vec{u}_1}{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}} \right\}$ is an orthogonal set.

$\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1(2) + 1(1) = -3+2+1 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = 3(-1) + 1(-4) + 1(7) = -3-4+7 = 0$

$\vec{u}_2 \cdot \vec{u}_3 = (-1)(-1) + 2(-4) + 1(7) = 1 - 8 + 7 = 0$

The set is orthogonal.

Orthongal Basis

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis that is also an orthogonal set.

Theorem: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then each vector $\mathbf{y}$ in W can be written as the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p, \quad \text{where the weights}$$

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}.$$

Note

$$\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_{j-1}\vec{u}_{j-1} + c_j\vec{u}_j + c_{j+1}\vec{u}_{j+1} + \dots + c_p\vec{u}_p$$

$$\vec{y}\cdot\vec{u}_j = (c_1\vec{u}_1 + \dots + c_p\vec{u}_p)\cdot\vec{u}_j$$

$$= \underbrace{c_1\vec{u}_1\cdot\vec{u}_j}_{=0} + \underbrace{c_2\vec{u}_2\cdot\vec{u}_j}_{=0} + \dots + \underbrace{c_j u_j\cdot u_j}_{=c_j\|\vec{u}_j\|^2} + \dots + \underbrace{c_p\vec{u}_p\cdot\vec{u}_j}_{=0}$$

$$\Longrightarrow \quad c_j = \frac{\vec{y}\cdot\vec{u}_j}{\|\vec{u}_j\|^2}$$

Example

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$$ is an orthogonal basis of $\mathbb{R}^3$. Express

$\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3$

the vector $\mathbf{y} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of the basis vectors.

$\|\vec{u}_1\|^2 = 9+1+1 = 11, \quad \|\vec{u}_2\|^2 = 1+4+1 = 6, \quad \|\vec{u}_3\|^2 = 1+16+49 = 66$

$\vec{y} \cdot \vec{u}_1 = -2(3)+3(1)+0 = -3$

$\vec{y} \cdot \vec{u}_2 = -2(-1)+3(2)+0 = 8$

$\vec{y} \cdot \vec{u}_3 = -2(-1)+3(-4) = -10$

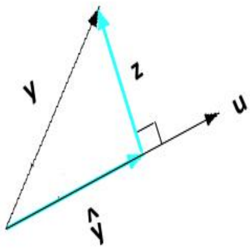
$$\vec{y} = \frac{-3}{11}\vec{u}_1 + \frac{8}{6}\vec{u}_2 - \frac{10}{66}\vec{u}_3$$

Projection

Given a nonzero vector **u**, suppose we wish to decompose another nonzero vector **y** into a sum of the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

in such a way that $\hat{\mathbf{y}}$ is parallel to **u** and **z** is perpendicular to **u**.



Projection

Since $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$, there is a scalar α such that

$$\hat{\mathbf{y}} = \alpha\mathbf{u}.$$

We need $\vec{y} = \hat{y} + \vec{z}$ with $\vec{z} \cdot \vec{u} = 0$

$$\vec{y} \cdot \vec{u} = (\hat{y} + \vec{z}) \cdot \vec{u} = \hat{y} \cdot \vec{u} + \vec{z} \cdot \vec{u}$$

$$= \hat{y} \cdot \vec{u} \qquad \text{setting } \hat{y} = \alpha\vec{u}$$

$$= (\alpha\vec{u}) \cdot \vec{u}$$

$$= \alpha(\vec{u} \cdot \vec{u})$$

$$\Rightarrow \quad \alpha = \frac{\vec{y} \cdot \vec{u}}{\|\vec{u}\|^2}$$

Projection onto the subspace $L = \text{Span}\{\mathbf{u}\}$

Notation: $\hat{\mathbf{y}} = \text{proj}_L \vec{y} = \left(\dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$

Example: Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is in Span$\{\mathbf{u}\}$ and $\mathbf{z}$ is orthogonal to $\mathbf{u}$.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{7\cdot 4 + 6\cdot 2}{4^2+2^2}\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{y} = \hat{y} + \vec{z} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Example Continued...

Determine the distance between the point $(7, 6)$ and the line Span$\{\mathbf{u}\}$.

The distance is $\|\vec{z}\|$.

$$\|\vec{z}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

Orthonormal Sets

Definition: A set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called an **orthonormal set** if it is an orthogonal set of **unit vectors.**

Definition: An **orthonormal basis** of a subspace W of $\mathbb{R}^n$ is a basis that is also an orthonormal set.

Example: Show that $\left\{ \underbrace{\begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix} -\frac{4}{5} \\ \\ \frac{3}{5} \end{bmatrix}}_{\vec{u}_2} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

$\vec{u}_1 \cdot \vec{u}_2 = \frac{3}{5}\left(\frac{-4}{5}\right) + \frac{4}{5}\left(\frac{3}{5}\right) = 0$

$\|\vec{u}_1\|^2 = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9+16}{25} = \frac{25}{25} = 1$

$\|\vec{u}_2\|^2 = \left(\frac{-4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16+9}{25} = 1$

The set is orthonormal.

To see that $\{\vec{u}_1, \vec{u}_2\}$ is a basis for $\mathbb{R}^2$,

note that $\det \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \left(\frac{3}{5}\right)^2 - \left(\frac{-4}{5}\right)\left(\frac{4}{5}\right) = 1$

The columns are lin. independent, hence they form a basis for $\mathbb{R}^2$.

Hence the set is an orthonormal basis for $\mathbb{R}^2$.

Orthogonal Matrix

Consider the matrix $U = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ whose columns are the vectors in the last example. Compute the product

$$U^T U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that $U^T = U^{-1}$

What does this say about U^{-1}?

Orthogonal Matrix

Definition: A square matrix U is called an **orthogonal matrix** if $U^T = U^{-1}$.

Theorem: An $n \times n$ matrix U is orthogonal if and only if it's columns form an orthonormal basis of $\mathbb{R}^n$.

The linear transformation associated to an orthogonal matrix preserves *lenghts* and *angles* in the following sense:

Theorem: Orthogonal Matrices

Let U be an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ vectors in $\mathbb{R}^n$. Then

(a) $\|U\mathbf{x}\| = \|\mathbf{x}\|$

(b) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, in particular

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof (of (a)):

$$\|U\vec{x}\|^2 = (U\vec{x}) \cdot (U\vec{x})$$

By definition of the dot product.

$$= (U\vec{x})^T (U\vec{x})$$

$$= (\vec{x}^T U^T)(U\vec{x})$$

Since

$$(AB)^T = B^T A^T$$

$$= \vec{x}^T (U^T U) \vec{x}$$

$$= \vec{x}^T I \vec{x}$$

$$= \vec{x}^T \vec{x} = \|\vec{x}\|^2$$

Hence $\|U\vec{x}\| = \|\vec{x}\|$