Oct. 26 Math 1190 sec. 51 Fall 2016

Section 4.5: Indeterminate Forms & L'Hôpital's Rule

Consider the following three limit statements (all of which are true):

(a) $$\lim_{x\to 1}\frac{x^2-1}{x-1}=2$$

(b) $$\lim_{x\to 0}\frac{\sin x}{x}=1$$

(c) $$\lim_{x\to 3}\frac{x^2-9}{(x-3)^2}\quad \text{doesn't exist}$$

Note: Each of these three limits involve both numerator and denominator going to zero—giving the form $\frac{0}{0}$. In the top two, the limit exists, but the limits are different. In the third, the limit doesn't exist.

Indeterminate Forms

$0/0$ is called an **Indeterminate form**.

Other indeterminate forms we'll encounter include

$$\frac{\pm\infty}{\pm\infty}, \quad \infty - \infty, \quad 0\infty, \quad 1^{\infty}, \quad 0^{0}, \quad \text{and} \quad \infty^{0}.$$

Indeterminate forms are not defined (as numbers)

Theorem: l'Hospital's Rule

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0$$

OR if

$$\lim_{x \to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \pm\infty$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Evaluate each limit if possible

(a) $\lim_{x \to 1} \frac{\ln x}{x - 1}$ $=$ "$\frac{0}{0}$"

Note : $\lim_{x \to 1} \ln x = \ln 1 = 0$

and $\lim_{x \to 1} x - 1 = 1 - 1 = 0$

apply l'H rule

$= \lim_{x \to 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx}(x-1)}$

$= \lim_{x \to 1} \frac{\frac{1}{x}}{1-0} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \frac{\frac{1}{1}}{1} = 1$

(b) $\lim_{x\to\infty} xe^{-x}$ $=$ "$\infty \cdot 0$"

$$= \lim_{x\to\infty} \frac{x}{e^x} = \text{"}\frac{\infty}{\infty}\text{"}$$

apply l'H rule

$$= \lim_{x\to\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx} e^x}$$

$$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty}$$

$$= 0$$

$$\lim_{x\to\infty} x = \infty$$

$$\lim_{x\to\infty} e^{-x} = 0$$

Options:

write $xe^{-x} = \frac{x}{\frac{1}{e^{-x}}} = \frac{x}{e^x}$

or

$xe^{-x} = \frac{e^{-x}}{\frac{1}{x}}$

(c) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ $=$ "$\dfrac{0}{0}$"

$\lim_{x\to 0} 1-\cos x = 1-\cos 0$

$= 1 - 1 = 0$

apply l'H rule $= \lim_{x\to 0} \dfrac{\sin x}{2x} =$ "$\dfrac{0}{0}$"

apply l'H again $= \lim_{x\to 0} \dfrac{\cos x}{2} = \dfrac{\cos 0}{2}$

$= \dfrac{1}{2}$

Questions

(1) (**True or False**) The limit $\lim_{x\to 3} \frac{2x-6}{\ln(x/3)}$ gives the indeterminate form "0/0".

$\lim_{x\to 3} 2x-6 = 6-6=0$

$\lim_{x\to 3} \ln\left(\frac{x}{3}\right) = \ln\left(\frac{3}{3}\right) = \ln 1 = 0$

(2) Using L'Hôpital's rule, $\lim_{x\to 3} \frac{2x-6}{\ln(x/3)} =$ $\lim_{x\to 3} \frac{2x-6}{\ln x - \ln 3}$

(a) L'Hôpital's rule doesn't apply since there is no indeterminate form.

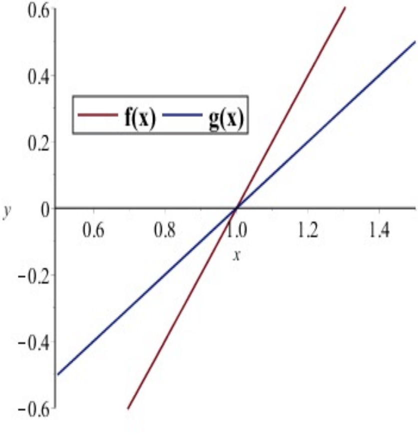
(b) $\frac{2}{3}$

(c) 6

(d) the limit doesn't exist.

$\lim_{x\to 3} \frac{2}{\frac{1}{x} - 0} = \frac{2}{\frac{1}{3}} = 2 \cdot 3 = 6$

Question



$y = f(x)$ and $y = g(x)$ close to $x = 1$ are plotted on the same set of axes. Note that

$$\lim_{x\to 1} f(x) = 0 \quad \text{and} \quad \lim_{x\to 1} g(x) = 0$$

From the graph, only one of the following limit statements could be true. Which one?

(a) $\lim_{x\to 1} \frac{f(x)}{g(x)} = 0$

(b) $\lim_{x\to 1} \frac{f(x)}{g(x)} = 2$

(c) $\lim_{x\to 1} \frac{f(x)}{g(x)} = -2$

Think ratio of slopes.

The $\frac{\text{slope of } f}{\text{slope of } g}$ near $c=1$ would have to be positive.

l'Hospital's Rule is not a "Fix-all"

Evaluate $\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \text{"}\frac{\infty}{\infty}\text{"}$ Use l'H rule

$= \lim_{x\to 0^+} \frac{-\csc^2 x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{\csc x}{\cot x} = \text{"}\frac{\infty}{\infty}\text{"}$ Use l'H again

$= \lim_{x\to 0^+} \frac{-\csc x \cot x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{\cot x}{\csc x}$ Dead end.

$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\cos x}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{1} = \lim_{x\to 0^+} \cos x = 1$

Don't apply it if it doesn't apply!

$$\lim_{x \to 2} \frac{x+4}{x^2-3} = \frac{6}{1} = 6$$

Correct

BUT

$$\lim_{x \to 2} \frac{\frac{d}{dx}(x+4)}{\frac{d}{dx}(x^2-3)} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$$

nonsense

Remarks:

- l'Hopital's rule only applies directly to the forms $0/0$, or $(\pm\infty)/(\pm\infty)$.
- Multiple applications may be needed, or it may not result in a solution.
- It can be applied indirectly to the form $0 \cdot \infty$ by turning the product into a quotient.
- Derivatives of numerator and denominator are taken **separately**–this is NOT a *quotient rule* application.
- Applying it where it doesn't belong likely produces nonsense!

The form $\infty - \infty$

Evaluate the limit if possible

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \text{"}\infty - \infty\text{"}$$

$$\lim_{x \to 1^+} \frac{1}{\ln x} = \frac{1}{0} = \infty$$

$$\lim_{x \to 1^+} \frac{1}{x-1} = \frac{1}{0} = \infty$$

$$= \lim_{x \to 1^+} \left(\frac{x-1}{\ln x \, (x-1)} - \frac{\ln x}{\ln x \, (x-1)} \right)$$

$$= \lim_{x \to 1^+} \frac{x-1-\ln x}{(\ln x)(x-1)} = \text{"}\frac{0}{0}\text{"}$$

Use l'H rule

$$= \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + (\ln x) \cdot 1} = \lim_{x \to 1^+} \left(\frac{1 - \frac{1}{x}}{1 - \frac{1}{x} + \ln x} \right) \cdot \frac{x}{x}$$

$$= \lim_{x \to 1^+} \frac{x - 1}{x - 1 + x\ln x} = \text{"}\frac{0}{0}\text{"}$$

Use l'H rule again

$$= \lim_{x \to 1^+} \frac{1}{1 + 1 \cdot \ln x + x \cdot \frac{1}{x}}$$

$$= \frac{1}{1 + \ln 1 + 1 \cdot \frac{1}{1}} = \frac{1}{1 + 0 + 1} = \frac{1}{2}$$

Indeterminate Forms 1^∞, 0^0, and ∞^0

Since the logarithm and exponential functions are continuous, and $\ln(x^r) = r \ln x$, we have

$$\lim_{x \to a} F(x) = \exp\left(\ln\left[\lim_{x \to a} F(x)\right]\right) = \exp\left(\lim_{x \to a} \ln F(x)\right)$$

provided this limit exists.

To take $\lim_{x \to a} F(x)$, take

$\lim_{x \to a} \ln(F(x))$. If $\lim_{x \to a} \ln(F(x)) = L$

then $\lim_{x \to a} F(x) = e^L$

Use this property to show that

$$\lim_{x\to 0^+}(1+x)^{1/x} = e$$

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \text{"} 1^{\infty} \text{"}$$

Consider $\ln (1+x)^{\frac{1}{x}}$.

$$\lim_{x\to 0^+} \ln (1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \text{"} \frac{0}{0} \text{"}$$

Use l'H rule

$$= \lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$= \lim_{x \to 0^+} \frac{1}{1+x} = \frac{1}{1+0} = 1$$

So $\displaystyle \lim_{x \to 0^+} \ln (1+x)^{\frac{1}{x}} = 1$

Hence $\displaystyle \lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = e^1 = e$

Evaluate

$\lim_{x\to0^+} x^x$ $=$ "0^0"

Take $\lim_{x\to0^+} \ln(x^x) = \lim_{x\to0^+} x\ln x$

$=$ "$0\cdot(-\infty)$"

Options $x\ln x = \dfrac{\ln x}{\frac{1}{x}}$ or $x\ln x = \dfrac{x}{\frac{1}{\ln x}}$

$\lim_{x\to0^+} x\ln x = \lim_{x\to0^+} \dfrac{\ln x}{\frac{1}{x}} =$ "$\dfrac{-\infty}{\infty}$"

Use l'H rule

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x \to 0^+} \frac{1}{\cancel{x}} \left(\frac{-x^{\cancel{2}}}{1} \right)$$

$$= \lim_{x \to 0^+} -x = -0 = 0$$

$$\lim_{x \to 0^+} \ln\left(x^x\right) = 0 \quad \text{so} \quad \lim_{x \to 0^+} x^x = e^0 = 1$$

Question

(a) **True or False:** Since $1^n = 1$ for every integer n, we should conclude that the indeterminate form 1^∞ is equal to 1.

(b) **True or False:** Even though $a^0 = 1$ for every nonzero number a, we can not conclude that the indeterminate form 0^0 is equal to 1.