Oct. 26 Math 1190 sec. 52 Fall 2016

Section 4.5: Indeterminate Forms & L'Hôpital's Rule

$0/0$ is called an **Indeterminate form**.

Other indeterminate forms we'll encounter include

$$\frac{\pm\infty}{\pm\infty}, \quad \infty - \infty, \quad 0\infty, \quad 1^{\infty}, \quad 0^0, \quad \text{and} \quad \infty^0.$$

Indeterminate forms are not defined (as numbers)

Theorem: l'Hospital's Rule

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0$$

OR if

$$\lim_{x \to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \pm\infty$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Evaluate each limit if possible

(a) $\lim_{x \to 0} \dfrac{\sin(3x)}{x}$ $=$ "$\dfrac{0}{0}$"

$\lim_{x \to 0} \sin(3x) = \sin(3 \cdot 0) = 0$

$\lim_{x \to 0} x = 0$

Apply l'H rule

$= \lim_{x \to 0} \dfrac{\frac{d}{dx} \sin(3x)}{\frac{d}{dx} x}$

$= \lim_{x \to 0} \dfrac{\cos(3x) \cdot 3}{1} = \dfrac{\cos(0) \cdot 3}{1} = \dfrac{3 \cdot 1}{1} = 3$

(b) $\lim_{x\to\infty} xe^{-x}$ $=$ "$\infty \cdot 0$"

$\lim_{x\to\infty} x = \infty$

$\lim_{x\to\infty} e^{-x} = 0$

$= \lim_{x\to\infty} \frac{x}{e^x} =$ "$\frac{\infty}{\infty}$"

Options:

write

$xe^{-x} = \frac{x}{\frac{1}{e^{-x}}} = \frac{x}{e^x}$

or

$xe^{-x} = \frac{e^{-x}}{\frac{1}{x}}$

Use l'H rule

$= \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x}$

$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty}$

$= 0$

(c) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ $= "\frac{0}{0}"$

$\lim_{x \to 0} 1 - \cos x = 1 - \cos 0$
$= 1 - 1 = 0$

Use l'H rule $= \lim_{x \to 0} \frac{\sin x}{2x} = "\frac{0}{0}"$

Use l'H rule again $= \lim_{x \to 0} \frac{\cos x}{2}$

$= \frac{\cos 0}{2} = \frac{1}{2}$

Questions

(1) (**True** or **False**) The limit $\lim_{x\to 3} \frac{2x-6}{\ln(x/3)}$ gives the indeterminate form "0/0".

$\lim_{x\to 3} 2x-6 = 6-6 = 0$

$\lim_{x\to 3} \ln\left(\frac{x}{3}\right) = \lim_{x\to 3} (\ln x - \ln 3)$

$= \ln 3 - \ln 3 = 0$

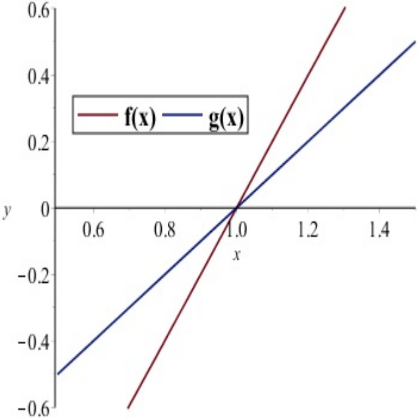
(2) Using L'Hôpital's rule, $\lim_{x\to 3} \frac{2x-6}{\ln(x/3)} =$

(a) L'Hôpital's rule doesn't apply since there is no indeterminate form.
(b) $\frac{2}{3}$
(c) 6
(d) the limit doesn't exist.

$\lim_{x\to 3} \frac{2x-6}{\ln x - \ln 3} = \lim_{x\to 3} \frac{2}{\frac{1}{x}}$

$= \frac{2}{\frac{1}{3}} = 2\cdot 3 = 6$

Question



$y = f(x)$ and $y = g(x)$ close to $x = 1$ are plotted on the same set of axes. Note that

$$\lim_{x\to 1} f(x) = 0 \quad \text{and} \quad \lim_{x\to 1} g(x) = 0$$

From the graph, only one of the following limit statements could be true. Which one?

(a) $\displaystyle\lim_{x\to 1} \frac{f(x)}{g(x)} = 0$

(b) $\displaystyle\lim_{x\to 1} \frac{f(x)}{g(x)} = 2$

(c) $\displaystyle\lim_{x\to 1} \frac{f(x)}{g(x)} = -2$

Think ratio of slopes

$$\frac{\text{Slope of } f}{\text{Slope of } g}$$

Both are positive and the slope of f is bigger than the slope of g.

l'Hospital's Rule is not a "Fix-all"

Evaluate $\lim_{x \to 0^+} \dfrac{\cot x}{\csc x}$ $=$ "$\dfrac{\infty}{\infty}$" Use l'H rule

$= \lim_{x \to 0^+} \dfrac{-\csc^2 x}{-\csc x \cot x} = \lim_{x \to 0^+} \dfrac{\csc x}{\cot x} =$ "$\dfrac{\infty}{\infty}$" Use l'H again

$= \lim_{x \to 0^+} \dfrac{-\csc x \cot x}{-\csc^2 x} = \lim_{x \to 0^+} \dfrac{\cot x}{\csc x}$ Dead end

Trig IDs

$\lim_{x \to 0^+} \dfrac{\cot x}{\csc x} = \lim_{x \to 0^+} \dfrac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \lim_{x \to 0^+} \dfrac{\cos x}{\cancel{\sin x}} \cdot \dfrac{\cancel{\sin x}}{1} = \lim_{x \to 0^+} \cos x = \cos 0$

$= 1$

Don't apply it if it doesn't apply!

$$\lim_{x \to 2} \frac{x+4}{x^2-3} = \frac{6}{1} = 6$$

BUT

$$\lim_{x \to 2} \frac{\frac{d}{dx}(x+4)}{\frac{d}{dx}(x^2-3)} \quad = \quad \lim_{x \to 2} \frac{1}{2x} \quad = \quad \frac{1}{4}$$

Remarks:

- l'Hopital's rule only applies directly to the forms $0/0$, or $(\pm\infty)/(\pm\infty)$.

- Multiple applications may be needed, or it may not result in a solution.

- It can be applied indirectly to the form $0 \cdot \infty$ by turning the product into a quotient.

- Derivatives of numerator and denominator are taken **separately**–this is NOT a *quotient rule* application.

- Applying it where it doesn't belong likely produces nonsense!

The form $\infty - \infty$

Evaluate the limit if possible

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \text{"}\infty-\infty\text{"}$$

$$\lim_{x\to 1} \ln x = 0$$

$$\lim_{x\to 1} x-1 = 0$$

as $x \to 1^+$

$$\frac{1}{\ln x} - \frac{1}{x-1} \to \infty - \infty$$

as $x \to 1^-$

$$\frac{1}{\ln x} - \frac{1}{x-1} \to -\infty + \infty$$

$$= \lim_{x\to 1}\left(\frac{x-1}{(\ln x)(x-1)} - \frac{\ln x}{(\ln x)(x-1)}\right)$$

$$= \lim_{x\to 1} \frac{x-1-\ln x}{(\ln x)(x-1)} = \text{"}\frac{0}{0}\text{"}$$

Use l'H rule

$$= \lim_{x\to 1} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+(\ln x)\cdot 1}$$

$$= \lim_{x\to 1} \left(\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x} \right) \cdot \frac{x}{x}$$

$$= \lim_{x\to 1} \frac{x-1}{x-1+x\ln x} = \text{"} \frac{0}{0} \text{"}$$

use l'H rule again

$$= \lim_{x\to 1} \frac{1}{1+1\cdot \ln x + x\cdot \frac{1}{x}} = \frac{1}{1+\ln 1+1} = \frac{1}{2}$$

Indeterminate Forms 1^∞, 0^0, and ∞^0

Since the logarithm and exponential functions are continuous, and $\ln(x^r) = r \ln x$, we have

$$\lim_{x \to a} F(x) = \exp\left(\ln\left[\lim_{x \to a} F(x)\right]\right) = \exp\left(\lim_{x \to a} \ln F(x)\right)$$

provided this limit exists.

• we want $\lim_{x \to a} F(x)$, • take $\lim_{x \to a} \ln F(x)$

IF $\lim_{x \to a} \ln F(x) = L$, then $\lim_{x \to a} F(x) = e^L$

Use this property to show that

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

As $x \to 0$ $\quad 1+x \to 1$

and $\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \text{"}1^{\infty}\text{"} \quad \text{and} \quad \lim_{x\to 0^-} (1+x)^{\frac{1}{x}} = \text{"}1^{-\infty}\text{"} = 1^{\infty}\text{"}$$

Lets take $\lim_{x\to 0^+} \ln (1+x)^{\frac{1}{x}}$

$$\lim_{x\to 0} \ln (1+x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{1}{x} \ln (1+x)$$

$$= \lim_{x \to 0} \frac{\ln(1+x)}{x} = \text{"}\frac{0}{0}\text{"}$$

Use l'H rule

$$= \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = \frac{\frac{1}{1+0}}{1} = 1$$

We found $\lim_{x \to 0} \ln(1+x)^{\frac{1}{x}} = 1$

so $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e^{1} = e$

Evaluate

$$\lim_{x\to 0^+} x^x$$

$= \text{"}0^0\text{"}$

we'll take $\lim\limits_{x\to 0^+} \ln x^x = \lim\limits_{x\to 0^+} x \ln x$

Options $x \ln x = \dfrac{\ln x}{\frac{1}{x}}$ or $x \ln x = \dfrac{x}{\frac{1}{\ln x}}$

$= \dfrac{\ln x}{x^{-1}}$

$\lim\limits_{x\to 0^+} x \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \text{"}\dfrac{-\infty}{\infty}\text{"}$ use l'H rule

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-x^{-2}}$$

$$= \lim_{x \to 0^+} \cancel{\frac{1}{x}}\left(-x^{\cancel{2}}\right) = \lim_{x \to 0^+} -x = 0$$

$$\lim_{x \to 0^+} \ln x^x = 0 \quad \text{so} \quad \lim_{x \to 0^+} x^x = e^0 = 1$$