October 26 Math 2306 sec. 53 Fall 2018

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

we'll use a partial fraction decomp.

$$\frac{s-8}{s^2-2s} = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$s-8 = A(s-2) + Bs$$

set $s=0$

$$0-8 = A(-2) + B\cdot 0 \Rightarrow -8 = -2A \Rightarrow A = 4$$

set $s=2$

$$2-8 = A(0) + B(2) \Rightarrow -6 = 2B \Rightarrow B = -3$$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4\cdot 1 - 3e^{2t}$$

$\frac{1}{s-a}$ for $a=2$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

By definition $\quad \mathscr{L}\left\{e^t t^2\right\}=\displaystyle\int_0^\infty e^{-st}e^t t^2\,dt$

Note: $e^{-st}\cdot e^{t} = e^{-st+t} = e^{-(s-1)t}$

$$= \int_0^\infty e^{-(s-1)t}\, t^2\, dt$$

$$\frac{2}{w^3} \xleftarrow{\text{from the table}} = \int_0^\infty e^{-wt}\, t^2\, dt \quad \text{if } w=s-1$$

Observe that this is simply the Laplace transform of $f(t)=t^2$ evaluated at $s-1$. Letting $F(s)=\mathscr{L}\left\{t^2\right\}$, we have

$$F(s-1)=\frac{2}{(s-1)^3}.$$

Theorem (translation in s)

Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

Q: Does the denominator factor?

If yes - do partial fractions

If no - complete the square

s^2+2s+2 is irreducible

$s^2+2s+1-1+2 = (s+1)^2+1$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}$$

Note $s = s+1-1$

$$= \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$k=1$ $a=-1$ $\qquad$ $k=1$ $a=-1$

$$= e^{-t}\cos t - e^{-t}\sin t$$

From table: $\mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$

$\mathcal{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$ Partial fractions

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \qquad s(s-1)^2$$

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$= (A+B)s^2 + (-2A-B+C)s + A$$

$$A=1$$

$$-2A-B+C=3 \qquad C=3+B+2A=3-2+2=3$$

$$A+B=-1 \quad \Rightarrow \quad B=-1-A=-2$$

$$\mathcal{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{2}{s-1}+\frac{3}{(s-1)^2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

looks like

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$= 1 - 2e^t + 3e^t t$

$F(s-1) = \dfrac{1}{(s-1)^2}$ what is $F(s)$?