October 26 Math 2306 sec 54 Fall 2015

Section 7.1: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} \cdot 2t\,dt + \int_{10}^\infty e^{-st} \cdot 0\,dt$$

$$= \int_0^{10} 2e^{-st}\, t\, dt$$

For $s \neq 0$, int by parts

$u = 2t \quad du = 2dt$

$v = \frac{-1}{s}e^{-st} \quad dv = e^{-st}dt$

$$= -\frac{2t}{s}e^{-st}\Big|_0^{10} + \frac{2}{s}\int_0^{10} e^{-st}\, dt$$

$$= \frac{-20}{s}e^{-10s} - 0 + \frac{2}{s}\left[\frac{-1}{s}e^{-st}\Big|_0^{10}\right.$$

$$= \frac{-20}{s}e^{-10s} + \frac{2}{s}\left[\frac{-1}{s}e^{-10s} - \frac{-1}{s}e^{0}\right]$$

$$= \frac{2}{s^2} - \frac{2}{s^2}e^{-10s} - \frac{20}{s}e^{-10s}$$

for $s \neq 0$

If $\quad s=0$

$$\mathcal{L}\{f(t)\} = \int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Use the Table to Evaluate $\mathscr{L}\{f(t)\}$

(a) $f(t) = 2t^2 + e^{6t}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2t^2 + e^{6t}\}$$

$$= 2\mathcal{L}\{t^2\} + \mathcal{L}\{e^{6t}\}$$

$$= 2\left(\frac{2!}{s^3}\right) + \frac{1}{s-6} = \frac{4}{s^3} + \frac{1}{s-6}, \quad s>6$$

$s>0$ $\quad$ $s>6$

(b) $f(t) = \cos 2t - \sin 3t$

$$\mathcal{L}\{\cos 2t - \sin 3t\} = \mathcal{L}\{\cos 2t\} - \mathcal{L}\{\sin 3t\}$$

$$= \frac{s}{s^2+2^2} - \frac{3}{s^2+3^2} = \frac{s}{s^2+4} - \frac{3}{s^2+9}, \quad s>0$$

$s>0$

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$$\mathcal{L}\{4 - 4t + t^2\} = 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1}{s^2}\right) + \frac{2}{s^3} = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

$s > 0$

(d) $f(t) = \sin^2 5t = \frac{1}{2} - \frac{1}{2}\cos(2 \cdot 5t) = \frac{1}{2} - \frac{1}{2}\cos(10t)$

$$\mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(10t)\right\} = \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos(10t)\}$$

$$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2}\frac{s}{s^2+10^2} = \frac{1}{2s} - \frac{s}{2(s^2+100)}$$

$s > 0$

Hyperbolic Sine and Cosine

Define the hyperbolic sine and cosine functions

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \text{and} \quad \cosh(t) = \frac{e^t + e^{-t}}{2}, \quad \text{respectively.}$$

Find the Laplace transforms of sinh t and cosh t.

$$\mathcal{L}\{\sinh t\} = \mathcal{L}\left\{\tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t}\right\}$$

$$= \tfrac{1}{2}\mathcal{L}\{e^t\} - \tfrac{1}{2}\mathcal{L}\{e^{-t}\}$$

$$= \tfrac{1}{2}\frac{1}{s-1} - \tfrac{1}{2}\frac{1}{s+1} = \tfrac{1}{2}\left(\frac{s+1 - (s-1)}{(s+1)(s-1)}\right)$$

$s > 1$ $\qquad$ $s > -1$

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}, \quad s>1$$

$$\mathcal{L}\{\cosh t\} = \mathcal{L}\left\{\frac{1}{2}e^{t} + \frac{1}{2}e^{-t}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{t}\} + \frac{1}{2}\mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{2}\underbrace{\frac{1}{s-1}}_{s>1} + \frac{1}{2}\underbrace{\frac{1}{s+1}}_{s>-1} = \frac{1}{2}\left(\frac{s+1+s-1}{(s+1)(s-1)}\right)$$

$$= \frac{s}{s^2-1}, \quad s>1$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

some functions don't have a Laplace transform.

e.g. $f(t) = \frac{1}{t}$

Section 7.2: Inverse Transforms and Derivatives

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \Longleftrightarrow \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ the inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets {} **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

From the table

$$\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = t^6$$

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

Note $\frac{1}{s^7} = \frac{1}{6!}\frac{6!}{s^7}$

So $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\frac{6!}{s^7}\right\} = \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}t^6$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+3^2} + \dfrac{1}{s^2+3^2}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\,\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos 3t + \frac{1}{3}\sin 3t$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s-8}{s(s-2)}\right\}$

Use a partial fraction decomp

$$\frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Mult. both sides by $s(s-2)$

$$s-8 = A(s-2) + Bs$$

set $s=0$ $\quad -8 = -2A \quad \Rightarrow \quad A = 4$

$s=2 \qquad -6 = 2B \quad \Rightarrow \quad B = -3$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4 - 3e^{2t}$$