October 26 Math 2306 sec. 56 Fall 2017

Section 15: Shift Theorems

Theorem: (Translation in s) Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

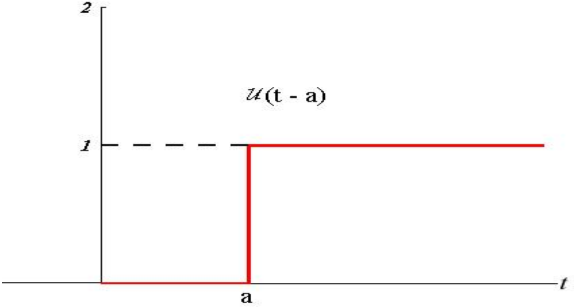

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

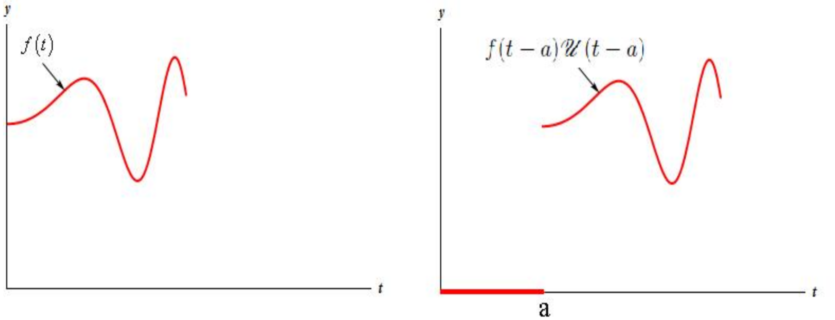

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

By definition

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathscr{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

diverges if $s=0$

Substitute

$\tau = t-a,\quad d\tau = dt$

$t = \tau + a$

when $t=a,\quad \tau = a-a = 0$

as $t\to\infty,\quad \tau\to\infty$

$$= \int_0^\infty e^{-s(\tau+a)}\,d\tau$$

Note $e^{-s(\tau+a)} = e^{-s\tau - sa} = e^{-s\tau} \cdot e^{-sa}$

$$= \int_0^\infty e^{-sa} e^{-s\tau}\, d\tau$$

$$= e^{-sa} \int_0^\infty e^{-s\tau}\, d\tau$$

$$= e^{-sa} \mathcal{L}\{1\} = e^{-s\tau} \cdot \frac{1}{s} = \frac{e^{-sa}}{s}$$

Example

Find the Laplace transform $\mathscr{L}\left\{h(t)\right\}$ where

$$h(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write h in terms of unit step functions.

$$h(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + \mathcal{U}(t-1)(-1+t)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Note that if $f(t)=t$, then

$$f(t-1)=t-1 \quad \text{so}$$

$$f(t-1)\mathcal{U}(t-1)=(t-1)\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $h(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{h\}$.

$$\mathscr{L}\{h(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$$* \quad \mathscr{L}\{t\} = \frac{1}{s^2}$$

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

Note $g(t) = g\left((t+a)-a\right)$

Example: Find $\mathscr{L}\left\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s}\mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$

$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\} = -e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1}\right)$

$= \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$

$\cos\left(t+\frac{\pi}{2}\right) = \cos t\cos\frac{\pi}{2} - \sin t\sin\frac{\pi}{2} = -\sin t$

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Here $f(t) = \mathscr{L}^{-1}\{F(s)\}$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

we need $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$ Use partial frac.

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow 1 = A(s+1) + Bs$$

set $s=0$ $A=1$

$s=-1$ $B=-1$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t}$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$$

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$. (Assume f is of exponential order c for some c.)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Let's int. by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

decays as $t \to \infty$

$$= 0 - f(0)e^0 + s\mathscr{L}\{f(t)\}$$

$$= s\mathscr{L}\{f(t)\} - f(0)$$

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\mathscr{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\}$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = \mathscr{L}\{g\}$$

Let

$\mathscr{L}\{y\} = Y(s)$

$\mathscr{L}\{g\} = G(s)$

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

$$as^2Y(s) - asy(0) - ay'(0) + bsY(s) - by(0) + cY(s) = G(s)$$

$$(as^2+bs+c)Y(s) - ay_0 s - ay_1 - by_0 = G(s)$$

$$(as^2+bs+c)Y(s) = ay_0 s + ay_1 + by_0 + G(s)$$

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$$

Letting

$$Q(s) = ay_0 s + ay_1 + by_0$$

and

$$P(s) = as^2+bs+c$$

$$y(t) = \mathcal{L}^{-1}\left\{ Y(s) \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)} \right\}$$

Solving IVPs

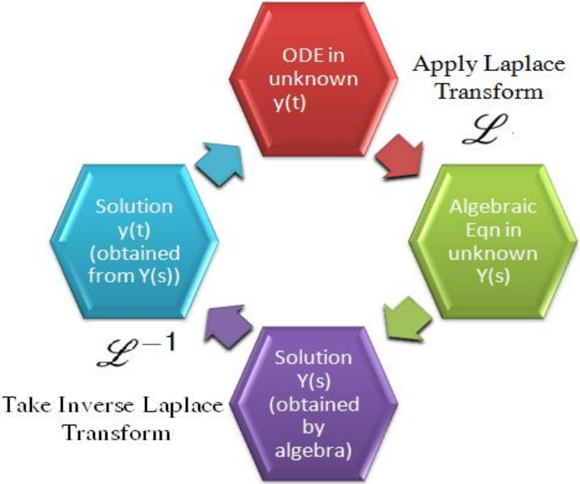

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**