October 28 Math 2306 sec 51 Fall 2015

Section 7.2: Inverse Transforms and Derivatives

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Solve the IVP using the Laplace Transform

$\mathcal{L}\{y\} = Y(s)$

(b) $y'' - 4y' = 6e^{3t} - 3e^{-t} \quad y(0) = 1, \quad y'(0) = -1$

$$\mathcal{L}\{y'' - 4y'\} = \mathcal{L}\{6e^{3t} - 3e^{-t}\}$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} = 6\mathcal{L}\{e^{3t}\} - 3\mathcal{L}\{e^{-t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4\left(sY(s) - y(0)\right) = \frac{6}{s-3} - \frac{3}{s+1}$$

$$(s^2 - 4s)Y(s) - s + 1 + 4 = \frac{6}{s-3} - \frac{3}{s+1}$$

$$(s^2 - 4s)Y(s) = \frac{6}{s-3} - \frac{3}{s+1} + s - 5$$

$$Y(s) = \frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)} + \frac{s-5}{s^2-4s}$$

Do partial fraction decomps

Mult by $(s-3)s(s-4)$

$$\frac{6}{(s-3)s(s-4)} = \frac{A}{s-3} + \frac{B}{s} + \frac{C}{s-4}$$

$$6 = As(s-4) + B(s-3)(s-4) + Cs(s-3)$$

Set $s=3$ $\quad 6 = -3A \Rightarrow A = -2$

$s=0$ $\quad 6 = 12B \Rightarrow B = \frac{1}{2}$

$s=4$ $\quad 6 = 4C \Rightarrow C = 3/2$

$$\frac{-3}{(s+1)s(s-4)} = \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s-4}$$

$$-3 = As(s-4) + B(s+1)(s-4) + Cs(s+1)$$

Set $s=-1$ $\quad -3 = 5A \implies A = -3/5$

$s=0$ $\quad -3 = -4B \implies B = 3/4$

$s=4$ $\quad -3 = 20C \implies C = -3/20$

$$\frac{s-5}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$$

$$s-5 = A(s-4) + Bs$$

Set $s=0$ $\quad -5 = -4A \implies A = 5/4$

$s=4$ $\quad -1 = 4B \implies B = -1/4$

$$Y(s) = \frac{-2}{s-3} + \frac{1/2}{s} + \frac{3/2}{s-4} + \frac{-3/5}{s+1} + \frac{3/4}{s} - \frac{3/20}{s-4} + \frac{5/4}{s} - \frac{1/4}{s-4}$$

$$Y(s) = \frac{-2}{s-3} + \frac{5/2}{s} + \frac{-3/5}{s+1} + \frac{11/10}{s-4}$$

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{Y(s)\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{-2}{s-3} + \frac{5/2}{s} - \frac{3/5}{s+1} + \frac{11/10}{s-4}\right\}\\
&= -2\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{5}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{11}{10}\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}
\end{aligned}$$

$$y(t) = -2e^{3t} + \frac{5}{2} - \frac{3}{5}e^{-t} + \frac{11}{10}e^{4t}$$

(c) $\frac{d^2y}{dt^2}+9y=e^t \quad y(0)=0, \quad y'(0)=0$

$$\mathcal{L}\{y''+9y\}=\mathcal{L}\{e^t\}$$

$$\mathcal{L}\{y''\}+9\mathcal{L}\{y\}=\mathcal{L}\{e^t\}$$

$$s^2Y(s)-s\cancel{y(0)}^{0}-\cancel{y'(0)}^{0}+9Y(s)=\frac{1}{s-1}$$

$$(s^2+9)Y(s)=\frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s^2+9)}$$

Partial Fraction Decomp

$$\frac{1}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$$

Clear fractions

$$1 = A(s^2+9) + (Bs+C)(s-1)$$

$$= As^2 + 9A + Bs^2 - Bs + Cs - C$$

$$0s^2 + 0s + 1 = (A+B)s^2 + (-B+C)s + 9A - C$$

$$\left.\begin{array}{l} A + B = 0 \\ -B + C = 0 \end{array}\right\} \overset{\text{add}}{\Rightarrow} \quad \begin{array}{r} A + C = 0 \\ 9A - C = 1 \\ \hline \text{add} \quad 10A = 1 \end{array} \Rightarrow A = \frac{1}{10}$$

$$9A \qquad - C = 1$$

$$B = -A = \frac{-1}{10}, \quad C = B = \frac{-1}{10}$$

$$Y(s) = \frac{1/10}{s-1} + \frac{\frac{-1}{10}s - \frac{1}{10}}{s^2+9} = \frac{1/10}{s-1} - \frac{1}{10}\frac{s}{s^2+9} - \frac{1}{10}\frac{1}{s^2+9}$$

$$Y(s) = \frac{1/10}{s-1} - \frac{1}{10}\,\frac{s}{s^2+3^2} - \frac{1}{30}\,\frac{3}{s^2+3^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1/10}{s-1} - \frac{1}{10}\,\frac{s}{s^2+3^2} - \frac{1}{30}\,\frac{3}{s^2+3^2}\right\}$$

$$= \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$\boxed{y(t) = \frac{1}{10}e^{t} - \frac{1}{10}\cos 3t - \frac{1}{30}\sin 3t}$$

Section 7.3: Translation Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$?

Consider the definition of $\mathscr{L}\left\{e^t t^2\right\}$

$$\mathscr{L}\{e^t t^2\} = \int_0^\infty e^{-st} e^t t^2 \, dt$$

Note

$$e^{-st} \cdot e^{t} = e^{-(s-1)t}$$

$$= \int_0^\infty e^{-(s-1)t} t^2 \, dt$$

This is the Laplace transform of t^2 evaluated @ $s-1$.

i.e. If $F(s) = \mathcal{L}\{t^2\}$ then

$$\mathcal{L}\{e^t t^2\} = F(s-1).$$

Since here, $F(s) = \dfrac{2}{s^3}$, $F(s-1) = \dfrac{2}{(s-1)^3}$

i.e. $\mathcal{L}\{e^t t^2\} = \dfrac{2}{(s-1)^3} \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)^3}\right\} = e^t t^2$