October 28 Math 2306 sec 54 Fall 2015

Section 7.2: Inverse Transforms and Derivatives

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \Longrightarrow \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ the inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

(d) $\mathscr{L}^{-1}\left\{\frac{(s+1)^2}{s^5}\right\} = \mathscr{L}^{-1}\left\{\frac{s^2+2s+1}{s^5}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s^2}{s^5} + \frac{2s}{s^5} + \frac{1}{s^5}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{s^4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{2!}\frac{2!}{s^3}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{3!}\frac{3!}{s^4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{4!}\frac{4!}{s^5}\right\}$$

$$= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \frac{1}{24}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\}$$

$$= \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{24}t^4$$

(e) $\mathscr{L}^{-1}\left\{\dfrac{2s^2+s+10}{s(s^2+5)}\right\}$

Partial fraction Decomp

$$\frac{2s^2+s+10}{s(s^2+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$$

Mult by $s(s^2+5)$

$$2s^2+s+10 = A(s^2+5)+(Bs+C)s$$

$$= As^2+5A+Bs^2+Cs$$

$$2s^2+s+10 = (A+B)s^2+Cs+5A$$

$5A = 10 \Rightarrow A = 2$

$C = 1$

$A + B = 2 \Rightarrow B = 2 - A = 0$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+s+10}{s(s^2+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{1}{s^2+5}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}\right\}$$

$$= 2 + \frac{1}{\sqrt{5}}\sin(\sqrt{5}\,t)$$

Transforms of Derivatives

Suppose f has a Laplace transform[1] and that f is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$.

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\, dt$$

Int. by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= 0 - e^{0} f(0) + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathscr{L}\{f(t)\}}$$

$\lim_{t\to\infty} e^{-st} f(t)$

[1] e.g. f is of exponential order c and assuming $s > c$.

$$\mathcal{L}\{f'(t)\} = -f(0) + s\,\mathcal{L}\{f(t)\}$$

If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Transforms of Derivatives

From $\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0)$, find

$$\mathscr{L}\left\{f''(t)\right\} = \int_0^\infty e^{-st} f''(t)\, dt$$

$$= s\,\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\mathscr{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\mathscr{L}\{f(t)\} - s f(0) - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$

Take the transform of both sides of the ODE

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\}$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = \mathscr{L}\{g(t)\}$$

$$a\left(s^2 Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + c\,Y(s) = G(s)$$

$$(as^2+bs+c)\,Y(s) - ay_0 s - ay_1 - by_0 = G(s)$$

Let's isolate $Y(s)$

$$(as^2+bs+c)\,Y(s) = ay_0 s + ay_1 + by_0 + G(s)$$

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$$

Solving IVPs

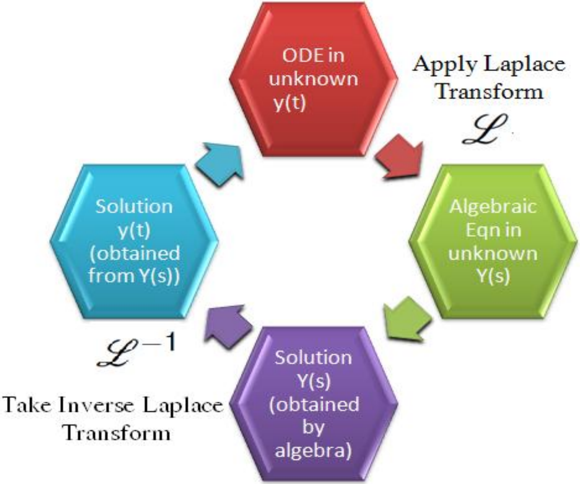

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

$Y(s) = \mathscr{L}\{y(t)\}$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$ is called the **zero input response**,

and

$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt} + 3y = 2t \quad y(0) = 2$

Let $Y(s) = \mathscr{L}\{y(t)\}$

$$\mathscr{L}\{y' + 3y\} = \mathscr{L}\{2t\}$$

$$\mathscr{L}\{y'\} + 3\mathscr{L}\{y\} = 2\mathscr{L}\{t\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{2}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3} = \frac{2+2s^2}{s^2(s+3)}$$

Use partial fraction decomp

$$\frac{2+2s^2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

Mult. by $s^2(s+3)$

$$0s + 2 + 2s^2 = As(s+3) + B(s+3) + Cs^2$$

$$= As^2 + 3As + Bs + 3B + Cs^2$$

$$= (A+C)s^2 + (3A+B)s + 3B$$

$$3B = 2 \implies B = \frac{2}{3}$$

$$3A + B = 0 \implies A = \frac{-B}{3} = \frac{-2}{9}$$

$$A + C = 2 \implies C = 2 - A = 2 + \frac{2}{9} = \frac{18+2}{9} = \frac{20}{9}$$

$$Y(s) = \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{20/9}{s+3}$$

$$y = \mathcal{L}^{-1}\left\{Y(s)\right\} = \frac{-2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$y = \frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}.$$