October 29 Math 2306 sec. 53 Fall 2018

Section 15: Shift Theorems

Theorem: (translation in s) Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Evaluate the Laplace Transform

(a) $\mathscr{L}\left\{e^{-2t}\cos(\pi t)\right\}$

$\mathscr{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$

$a = -2$

$$= \frac{s-(-2)}{(s-(-2))^2+\pi^2} = \frac{s+2}{(s+2)^2+\pi^2}$$

Evaluate the Inverse Laplace Transform

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+6)^4}\right\}$

Consider $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4}\right\}$

Looks like

$\dfrac{n!}{s^{n+1}}$ for $n=3$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{s^4}\right\}$$

$$= \frac{1}{3!}\,\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3!}t^3$$

$$= e^{-6t}\left(\frac{1}{3!}t^3\right)$$

$$= \frac{1}{6}t^3e^{-6t}$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Some people write

$u_a(t) = \mathscr{U}(t-a)$

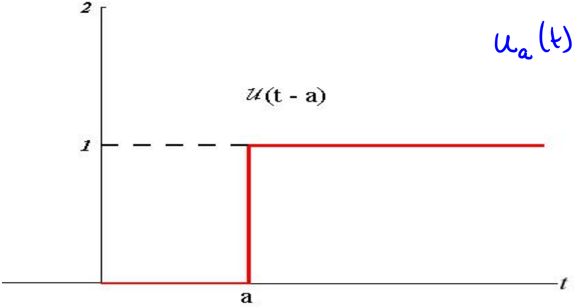

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Recall $\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$

If $0 \leq t < a$, then $\mathscr{U}(t-a) = 0$. Then

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot 0 + h(t)\cdot 0$$

$$= g(t)$$

For $t \geq a$, $\mathcal{U}(t-a) = 1$

$$g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) = g(t) - g(t)\cdot 1 + h(t)\cdot 1$$

$$= g(t) - g(t) + h(t)$$

$$= h(t)$$

So

$$g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases}$$

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

$$\mathscr{U}(t-2) = \begin{cases} 0, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}$$

$$\mathscr{U}(t-5) = \begin{cases} 0, & 0 \le t < 5 \\ 1, & t \ge 5 \end{cases}$$

$$f(t) = \underset{\text{on}}{e^t} - \underset{\text{off}}{e^t\mathscr{U}(t-2)} + \underset{\text{on}}{t^2\mathscr{U}(t-2)} - \underset{\text{off}}{t^2\mathscr{U}(t-5)} + \underset{\text{on}}{2t\,\mathscr{U}(t-5)}$$

Let's verify that this is correct.

For $0 \le t < 2$, $\mathscr{U}(t-2) = 0$ and $\mathscr{U}(t-5) = 0$

$$f(t) = e^t - e^t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + 2t \cdot 0 = e^t$$

For $2 \leq t < 5$ $\mathcal{U}(t-2) = 1$ and $\mathcal{U}(t-5) = 0$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + 2t \cdot 0 = t^2$$

For $t \geq 5$, $\mathcal{U}(t-2) = 1$ and $\mathcal{U}(t-5) = 1$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + 2t \cdot 1 = 2t$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

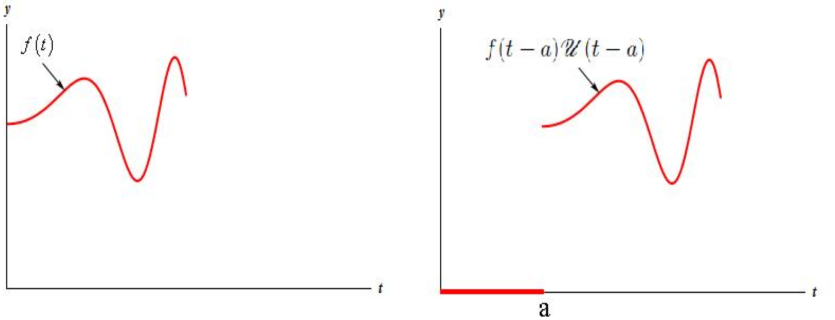

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

Note this is $e^{-as} \cdot \mathscr{L}\{1\}$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$$\mathcal{U}(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a\end{cases}$$

$$\mathcal{L}\{\mathcal{U}(t-a)\}=\int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$=\int_0^a \underbrace{e^{-st}\cdot 0}_{0}\,dt+\int_a^\infty e^{-st}\cdot 1\,dt$$

$$=\frac{-1}{s}e^{-st}\Big|_a^\infty \qquad \text{for } s>0$$

$$=\frac{-1}{s}\left(0-e^{-s\cdot a}\right)=\frac{1}{s}e^{-as}$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + \mathcal{U}(t-1)(-1+t)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$*$ If $g(t) = t$, then $g(t-1) = t-1$

and $\mathscr{L}\{t\} = \frac{1}{s^2}$

looks like $g(t-1)\mathscr{U}(t-1)$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{1}{s^2} \cdot e^{-1s} = \frac{1}{s} + \frac{e^{-s}}{s^2}$$