October 2 Math 2306 sec 51 Fall 2015

Section 4.4: Method of Undetermined Coefficients

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

Consider y_c: $\quad y'' - 2y' + 5y = 0$

$$m^2 - 2m + 5 = 0$$

$$m^2 - 2m + 1 + 4 = 0 \Rightarrow (m-1)^2 + 4 = 0$$

$$(m-1)^2 = -4$$

$$m - 1 = \pm 2i$$

$$m = 1 \pm 2i$$

$$y_1 = e^x \cos(2x), \quad y_2 = e^x \sin(2x) \Leftarrow \alpha = 1, \quad \beta = 2$$

So $y_c = C_1 e^x \cos(2x) + C_2 e^x \sin(2x)$

Let $g_1(x) = e^x$ and $g_2(x) = 7\sin(2x)$

$y_{p_1} = Ae^x$

works as written

$y_{p_2} = B\sin(2x) + C\cos(2x)$

works as written

Hence $y_p = Ae^x + B\sin(2x) + C\cos(2x)$.

Solve the IVP

$$y'' - 4y' + 4y = 8x - 4 \quad y(0) = 3, \quad y'(0) = -2$$

Find y_c: $\quad y'' - 4y' + 4y = 0 \qquad m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2$ repeated

$y_1 = e^{2x}, \quad y_2 = xe^{2x}$

and $\quad y_c = c_1 e^{2x} + c_2 x e^{2x}$

Find y_p: Use Undetermined Coefficients

$g(x) = 8x - 4 \qquad$ assume $\quad y_p = Ax + B$

$\leftarrow$ Doesn't solve the homogeneous eqn.

$$y_p = Ax + B, \quad y_p' = A, \quad y_p'' = 0$$

$$y_p'' - 4y_p' + 4y_p = 8x - 4$$

$$0 - 4(A) + 4(Ax + B) = 8x - 4$$

$$4Ax + (-4A + 4B) = 8x - 4$$

$$4A = 8 \Rightarrow A = 2$$

$$-4A + 4B = -4 \Rightarrow B = \frac{1}{4}(-4 + 4A) = \frac{4}{4} = 1$$

$$y_p = 2x + 1$$

The general solution to the ODE is

$$y = c_1 e^{2x} + c_2 x e^{2x} + 2x + 1$$

Now, impose $y(0) = 3$, $y'(0) = -2$.

$$y' = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x} + 2$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 e^0 + 2 \cdot 0 + 1 = 3$$

$$c_1 + 1 = 3 \implies c_1 = 2$$

$$y'(0) = 2c_1 e^0 + c_2 e^0 + 2c_2 \cdot 0 e^0 + 2 = -2$$

$$4 + c_2 + 2 = -2$$

$$c_2 = -2 - 2 - 4 = -8$$

$$\boxed{\text{The solution to the IVP is} \quad y = 2e^{2x} - 8xe^{2x} + 2x + 1.}$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find y_c: $\quad y'' - y = 0 \qquad m^2 - 1 = 0 \Rightarrow m^2 = 1, \quad m = \pm 1$

$m_1 = 1, \quad m_2 = -1$

$y_1 = e^{x}, \quad y_2 = e^{-x}$

$y_c = c_1 e^{x} + c_2 e^{-x}$

Find y_p: $\quad g(x) = 4e^{-x} \qquad y_p = (Ae^{-x})x = Axe^{-x}$

Solves the homogeneous eqn

This is correct..

$y_p = Axe^{-x}, \quad y_p' = Ae^{-x} - Axe^{-x}, \quad y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x}$

$$y_p'' - y_p = 4e^{-x}$$

$$-2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x} \Rightarrow -2A = 4 \Rightarrow A = -2$$

Hence $y_p = -2xe^{-x}$

The general solution to the ODE is

$$y = c_1 e^{x} + c_2 e^{-x} - 2xe^{-x}$$

Impose $y(0) = -1$, $y'(0) = 1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - 2 \cdot 0\, e^{0} = -1$$

$$c_1 + c_2 = -1$$

$$y'(0) = c_1 e^{0} - c_2 e^{0} - 2e^{0} + 2 \cdot 0\, e^{0} = 1$$

$$c_1 - c_2 - 2 = 1$$

$$c_1 + c_2 = -1$$

$$c_1 - c_2 = 3$$

add equations $2c_1 = 2 \Rightarrow c_1 = 1$

$$c_2 = -1 - c_1 = -1 - 1 = -2$$

The solution to the IVP is

$$\boxed{y = e^{x} - 2e^{-x} - 2xe^{-x}.}$$