October 30 Math 2306 sec 51 Fall 2015

Section 7.3: Translation Theorems

Theorem (translation in s) Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a).$$

e^{at}

$$\mathscr{L}\left\{e^{at} f(t)\right\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \mathscr{L}\left\{f(t)\right\}\big|_{s-a}$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at} t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

Evaluate the Laplace Transform

$\mathscr{L}\{t^4\} = \dfrac{4!}{s^5}$

(a) $\mathscr{L}\left\{e^{-3t}t^4\right\} = \dfrac{4!}{(s-(-3))^5}$

$= \dfrac{4!}{(s+3)^5}$

$\mathscr{L}\{\sin(\pi t)\} = \dfrac{\pi}{s^2+\pi^2}$

(b) $\mathscr{L}\left\{e^{2t}\sin(\pi t)\right\}$

$= \dfrac{\pi}{(s-2)^2+\pi^2}$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible

Complete the square

$s^2+2s+2 = s^2+2s+1+1$

$= (s+1)^2+1$

$= \mathscr{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}$

$s = s+1-1$

$= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$

$= e^{-t}\cos t - e^{-t}\sin t$

Using : $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Do a partial fraction decomp

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Mult by $s(s-1)^2$

$$1+3s-s^2 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$1+3s-s^2 = (A+B)s^2 + (-2A-B+C)s + A$$

$$\left.\begin{array}{rl} A + B & = -1 \\ -2A - B + C & = 3 \\ A & = 1 \end{array}\right\} \Rightarrow A = 1, \quad B = -1 - A = -2$$

$$C = 3 + B + 2A = 3 - 2 + 2 = 3$$

$$\mathcal{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 1 - 2e^{t} + 3te^{t}$$

since $\mathcal{L}\{t\} = \frac{1}{s^2}$

Solve the IVP using the Laplace Transform

$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$

Let $\mathcal{L}\{y\} = Y$

$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{te^{-2t}\}$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{te^{-2t}\}$

$s^2Y(s) - sy(0) - y'(0) + 4\left(sY(s) - y(0)\right) + 4Y(s) = \frac{1}{(s-(-2))^2}$

$(s^2 + 4s + 4)Y(s) - s \cdot 1 - 0 - 4 = \frac{1}{(s+2)^2}$

$$(s+2)^2 Y(s) = \frac{1}{(s+2)^2} + s + 4$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}$$

$$= \frac{1}{(s+2)^4} + \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2}$$

$$= \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= \frac{1}{6}\,\mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$y = \frac{1}{6}t^3 e^{-2t} + e^{-2t} + 2te^{-2t}$$