

## Section 7.2: Inverse Transforms and Derivatives

- ▶  $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- ▶  $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$
- ▶  $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0 \text{ for } n = 1, 2, \dots$
- ▶  $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- ▶  $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- ▶  $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

## Transforms of Derivatives

For  $y = y(t)$  defined on  $[0, \infty)$  having derivatives  $y'$ ,  $y''$  and so forth, if

$$\mathcal{L}\{y(t)\} = Y(s),$$

then

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2 Y(s) - sy(0) - y'(0),$$

⋮

$$\mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - y^{(n-1)}(0).$$

## Solve the IVP using the Laplace Transform

$$(b) \quad y'' - 4y' = 6e^{3t} - 3e^{-t} \quad y(0) = 1, \quad y'(0) = -1$$

Let  $\mathcal{L}\{y\} = Y(s)$

$$\mathcal{L}\{y'' - 4y'\} = \mathcal{L}\{6e^{3t} - 3e^{-t}\}$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} = 6\mathcal{L}\{e^{3t}\} - 3\mathcal{L}\{e^{-t}\}$$

$$s^2 Y(s) - s y(0) - y'(0) - 4(sY(s) - y(0)) = \frac{6}{s-3} - \frac{3}{s-(-1)}$$

$$(s^2 - 4s)Y(s) - s + 1 + 4 = \frac{6}{s-3} - \frac{3}{s+1}$$

$$Y(s) = \frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)} + \frac{s-5}{s^2-4s}$$

Partial fraction Decomp:

$$\frac{6}{(s-3)s(s-4)} = \frac{A}{s-3} + \frac{\beta}{s} + \frac{c}{s-4} \quad \text{Mult. by } (s-3)s(s-4)$$

$$6 = A s(s-4) + \beta(s-3)(s-4) + c s(s-3)$$

$$\text{Set } s=3 \quad 6 = -3A \Rightarrow A = -2$$

$$s=0 \quad 6 = 12\beta \Rightarrow \beta = \frac{1}{2}$$

$$s=4 \quad 6 = 4c \Rightarrow c = \frac{3}{2}$$

$$\frac{-3}{(s+1)s(s-4)} = \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s-4}$$

mult.  
 $(s+1)s(s-4)$

$$-3 = As(s-4) + B(s+1)(s-4) + Cs(s+1)$$

Set  $s = -1$        $-3 = 5A \Rightarrow A = \frac{-3}{5}$

$s = 0$        $-3 = -4B \Rightarrow B = \frac{3}{4}$

$s = 4$        $-3 = 20C \Rightarrow C = \frac{-3}{20}$

$$\frac{s-5}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$$

mult. by  
 $s(s-4)$

$$s-5 = A(s-4) + Bs$$

$$s=0 \quad -5 = -4A \Rightarrow A = \frac{5}{4}$$

$$s=4 \quad -1 = 4B \Rightarrow B = -\frac{1}{4}$$

$$Y(s) = \frac{-2}{s-3} + \frac{1_2}{s} + \frac{3_2}{s-4} + \frac{-3_1 s}{s+1} + \frac{3_4}{s} + \frac{-3_{20}}{s-4} + \frac{5_4}{s} - \frac{1_4}{s-4}$$

$$Y(s) = \frac{-2}{s-3} - \frac{3s}{s+1} + \frac{5s}{s} + \frac{11}{s-4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s-3} - \frac{3s}{s+1} + \frac{5s}{s} + \frac{11}{s-4}\right\}$$

$$= -2 \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{3}{5} \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{5}{2} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{11}{10} \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$y(t) = -2e^{3t} - \frac{3}{5}e^{-t} + \frac{5}{2} + \frac{11}{10}e^{4t}$$

$$(c) \frac{d^2y}{dt^2} + 9y = e^t \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{e^t\}$$

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{1}{s-1}$$

$$(s^2 + 9)Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s^2+9)}$$

Partial fraction decomp

$$\frac{1}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$$

Mult. by  
 $(s-1)(s^2+9)$

$$1 = A(s^2+9) + (Bs+C)(s-1)$$

$$= As^2 + 9A + Bs^2 - Bs + Cs - C$$

$$0s^2 + 0s + 1 = \underline{\underline{(A+B)s^2}} + \underline{\underline{(-B+C)s}} + \underline{\underline{9A-C}}$$

$$\begin{array}{l}
 A + B = 0 \\
 -B + C = 0 \\
 \hline
 9A - C = 1
 \end{array}
 \quad \left\{ \Rightarrow \begin{array}{l} A + C = 0 \\ 9A - C = 1 \end{array} \right. \quad \Rightarrow \quad A = \frac{1}{10}$$

$$B = -A = \frac{1}{10}, \quad C = B = \frac{1}{10}$$

$$Y(s) = \frac{\frac{1}{10}}{s-1} + \frac{\frac{1}{10}s - \frac{1}{10}}{s^2+9} = \frac{\frac{1}{10}}{s-1} - \frac{1}{10} \left( \frac{s}{s^2+9} + \frac{1}{s^2+9} \right)$$

$$Y(s) = \frac{1}{10} \frac{1}{s-1} - \frac{1}{10} \frac{s}{s^2+9} - \frac{1}{30} \frac{3}{s^2+9}$$

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\&= \frac{1}{10} \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{10} \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{1}{30} \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\end{aligned}$$

$$y(t) = \frac{1}{10} e^t - \frac{1}{10} \cos 3t - \frac{1}{30} \sin 3t$$