October 31 Math 2306 sec. 56 Fall 2017

Section 16: Laplace Transforms of Derivatives and IVPs

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Solving IVPs

- We'll start with a linear IVP (constant coefficient for now).

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

- We'll take the Laplace transform of both sides of the ODE.

$$\mathscr{L}\left\{ay'' + by' + cy\right\} = \mathscr{L}\left\{g(t)\right\}$$

- Letting $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$ use the various properties of Laplace transforms and necessary algebra to get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}, \quad (P \text{ and } Q \text{ are determined by the ODE})$$

- Then we find our solution y by taking the inverse transform

$$y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\}.$$

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt} + 3y = 2t \quad y(0) = 2$

Call $\mathcal{L}\{y\} = Y(s)$

$$\mathcal{L}\left\{\frac{dy}{dt} + 3y\right\} = \mathcal{L}\{2t\}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{2}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2} \Rightarrow (s+3)Y(s) = \frac{2}{s^2} + 2$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3}$$

We'll do a partial fraction decomp on $\dfrac{2}{s^2(s+3)}$

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$ Clear fractions

$$2 = As(s+3) + B(s+3) + Cs^2$$

$$= A(s^2+3s) + B(s+3) + Cs^2$$

$$0s^2 + 0s + 2 = (A+C)s^2 + (3A+B)s + 3B$$

$$A + C = 0 \qquad \Rightarrow \quad C = -A = \frac{2}{9}$$

$$3A + B = 0 \qquad \Rightarrow \quad A = -\frac{1}{3}B = \frac{-2}{9}$$

$$3B = 2 \qquad \Rightarrow \quad B = \frac{2}{3}$$

So $$Y(s) = \frac{-2/9}{s} + \frac{\frac{2}{3}}{s^2} + \frac{2/9}{s+3} + \frac{2}{s+3}$$

$$= \frac{\frac{-2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{\frac{20}{9}}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Our solution to the IVP

$$y = \mathcal{L}^{-1}\left\{\frac{\frac{-2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{\frac{20}{9}}{s+3}\right\}$$

$$y = \frac{-2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y = \frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}}$$

Let's verify that this solves the IVP.

Initial condition: $y(0) = \frac{-2}{9} + \frac{2}{3}\cdot 0 + \frac{20}{9}e^{0} = \frac{-2}{9} + \frac{20}{9}$

$$= \frac{18}{9} = 2$$

ODE: $y = \frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}$

$$y' = \frac{2}{3} + \frac{20}{9}e^{-3t}(-3) = \frac{2}{3} - \frac{20}{3}e^{-3t}$$

$$y' + 3y = \frac{2}{3} - \frac{20}{3}e^{-3t} + 3\left(\frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}\right)$$

$$= \cancel{\frac{2}{3}} - \cancel{\frac{20}{3}e^{-3t}} - \cancel{\frac{2}{3}} + 2t + \cancel{\frac{20}{3}e^{-3t}}$$

$$= 2t$$

So our y satisfies the ODE and the initial condition.

Solve the IVP using the Laplace Transform

$$y''+4y'+4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$$

$$\mathcal{L}\{y''+4y'+4y\} = \mathcal{L}\{te^{-2t}\}$$

$\mathcal{L}\{e^{at}f(t)\}$ with $a=-2$ and $f(t)=t$

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y'\}+4\mathcal{L}\{y\} = \mathcal{L}\{te^{-2t}\}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\Downarrow$$

$$\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$$

$$s^2Y(s)-s\underbrace{y(0)}_{1}-\underbrace{y'(0)}_{0}+4\left(sY(s)-\underbrace{y(0)}_{1}\right)+4Y(s) = \frac{1}{(s+2)^2}$$

$$s^2Y(s)-s+4sY(s)-4+4Y(s) = \frac{1}{(s+2)^2}$$

$$(s^2+4s+4)Y(s) = \frac{1}{(s+2)^2} + s + 4$$

$$(s+2)^2 Y(s) = \frac{1}{(s+2)^2} + s+4$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}$$

Note $s+4 = s+2+2$ so $\frac{s+4}{(s+2)^2} = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2}$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

Note $\frac{1}{(s+2)^2}$ looks like $\frac{1}{s^2}$ with s shifted by -2

$$\frac{1}{(s+2)^4} = \frac{1}{3!}\,\frac{3!}{(s+2)^4}$$ looks like $\frac{3!}{s^4}$ with

s shifted by -2.

$$Y(s) = \frac{1}{3!}\,\frac{3!}{(s+2)^4} + \frac{1}{s+2} + 2\,\frac{1}{(s+2)^2}$$

And $\quad y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{(s+2)^4} + \frac{1}{s+2} + 2\,\frac{1}{(s+2)^2}\right\}$$

$$y = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

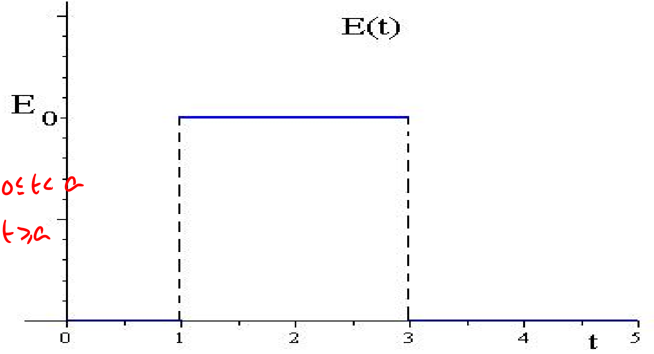
$$y = \frac{1}{6}t^3e^{-2t} + e^{-2t} + 2te^{-2t}$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$$L\frac{di}{dt} + Ri = E(t)$$

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$



$$E(t) = 0 - 0\,u(t-1) + E_0\,u(t-1) - E_0\,u(t-3) + 0\,u(t-3)$$

LR Circuit Example

The IVP is

$$\frac{di}{dt} + 10i = E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3), \quad i(0) = 0$$

$$\mathcal{L}\left\{\frac{di}{dt} + 10i\right\} = \mathcal{L}\left\{E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3)\right\}$$

$$\mathcal{L}\left\{\frac{di}{dt}\right\} + 10\mathcal{L}\{i\} = E_0\mathcal{L}\{\mathcal{U}(t-1)\} - E_0\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - \underbrace{i(0)}_{=0} + 10I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$(s+10)I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$I(s) = \frac{E_0 e^{-s}}{s(s+10)} - \frac{E_0 e^{-3s}}{s(s+10)}$$

Let's do a decomp on $\dfrac{1}{s(s+10)}$

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$1 = A(s+10) + Bs$$

Set $s=0$ $\quad 1 = 10A \implies A = \frac{1}{10}$

$s=-10$ $\quad 1 = -10B \implies B = \frac{-1}{10}$

$$I(s) = \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-s} - \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-3s}$$

$$\mathscr{L}^{-1}\left\{\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right\} = \frac{E_0}{10}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{E_0}{10}\mathscr{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$= \frac{E_0}{10} - \frac{E_0}{10}e^{-10t}$$

$$i(t) = \mathscr{L}^{-1}\{I(s)\}$$

$$= \left(\frac{E_0}{10} - \frac{E_0}{10}e^{-10(t-1)}\right)\mathcal{U}(t-1) - \left(\frac{E_0}{10} - \frac{E_0}{10}e^{-10(t-3)}\right)\mathcal{U}(t-3)$$

Let's write $i(t)$ in pieces

If $t<1$, $u(t-1)=0$ and $u(t-3)=0$

$$i(t)=0$$

If $1\le t<3$, $u(t-1)=1$ $u(t-3)=0$

$$i(t)=\frac{E_0}{10}-\frac{E_0}{10}e^{-10(t-1)}$$

If $t\geq 3$, $u(t-1)=1$ and $u(t-3)=1$

$$i(t)=\frac{E_0}{10}-\frac{E_0}{10}e^{-10(t-1)}-\frac{E_0}{10}+\frac{E_0}{10}e^{-10(t-3)}$$

$$i(t) = \frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)}$$

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}, & 1 \le t < 3 \\ \frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)}, & t \ge 3 \end{cases}$$