Section 6.4: Gram-Schmidt Orthogonalization

The goal here is to obtain an orthogonal basis for a vector space. The Gram-Schmidt process will allow us to generate an orthogonal basis if we start with an arbitrary basis.
Theorem: Gram Schmidt Process

Let \( \{x_1, \ldots, x_p\} \) be any basis for the nonzero subspace \( W \) of \( \mathbb{R}^n \).

Define the set of vectors \( \{v_1, \ldots, v_p\} \) via

\[
\begin{align*}
v_1 &= x_1 \\
v_2 &= x_2 - \left( \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \right) v_1 \\
v_3 &= x_3 - \left( \frac{x_3 \cdot v_1}{v_1 \cdot v_1} \right) v_1 - \left( \frac{x_3 \cdot v_2}{v_2 \cdot v_2} \right) v_2 \\
&\quad \vdots \\
v_p &= x_p - \sum_{j=1}^{p-1} \left( \frac{x_p \cdot v_j}{v_j \cdot v_j} \right) v_j.
\end{align*}
\]

Then \( \{v_1, \ldots, v_p\} \) is an orthogonal basis for \( W \). Moreover, for each \( k = 1, \ldots, p \)

\[
\text{Span}\{v_1, \ldots, v_k\} = \text{Span}\{x_1, \ldots, x_k\}.
\]
Example

Find an orthonormal (that’s *orthonormal* not just orthogonal) basis for \( \text{Col } A \) where

\[
A = \begin{bmatrix}
-1 & 6 & 6 \\
3 & -8 & 3 \\
1 & -2 & 6 \\
1 & -4 & -3 \\
\end{bmatrix}.
\]

We'll start with the columns of \( A \):

\[
\begin{align*}
\vec{x}_1 &= \begin{bmatrix}
-1 \\ 3 \\ 1 \\
1 \\
\end{bmatrix}, \\
\vec{x}_2 &= \begin{bmatrix}
6 \\ -8 \\ -2 \\ -9 \\
\end{bmatrix}, \\
\vec{x}_3 &= \begin{bmatrix}
6 \\ 3 \\ 6 \\ -3 \\
\end{bmatrix}.
\end{align*}
\]

\[
\vec{v}_1 = \vec{x}_1 = \begin{bmatrix}
3 \\ 0 \\ 0 \\
\end{bmatrix}.
\]
\[ \mathbf{V}_2 = \mathbf{y}_2 - \frac{\mathbf{\hat{x}}_2 \cdot \mathbf{V}_1}{\|\mathbf{V}_1\|^2} \mathbf{V}_1 \]

\[ \mathbf{\hat{x}}_2 \cdot \mathbf{V}_1 = -6 - 2y - 2 - y = -36 \]

\[ \|\mathbf{V}_1\|^2 = 1 + 9 + 1 + 1 = 12 \]

\[ \mathbf{V}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -y \end{bmatrix} - \left( \frac{-36}{12} \right) \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \]

\[ = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \]
\[ \vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \vec{v}_2 \]

\[ \vec{x}_3 \cdot \vec{v}_1 = -6 + 9 + 6 - 3 = 6 \quad ||\vec{v}_1||^2 = 12 \]

\[ \vec{x}_3 \cdot \vec{v}_2 = 18 + 3 + 6 + 3 = 30 \quad ||\vec{v}_2||^2 = 12 \]

\[ \vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \]
\[ \sqrt{3} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 15/2 \\ 5/2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} 6-7 \\ 3-4 \\ 6-3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \]

\[ \|\mathbf{v}_3\|^2 = 1+1+9+1 = 12 \]

\[ \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = \sqrt{12} \]

We have an orthogonal basis

\[ \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \right\} \]
An orthonormal basis is

\[
\left\{ \begin{bmatrix} \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{12}} \\ \frac{-1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix} \right\}
\]
Section 6.7 Inner Product Spaces

Definition: An inner product on a vector space $V$ is a function which assigns to each pair of vectors $u$ and $v$ in $V$ a real number denoted by $<u, v>$ and that satisfies the following four axioms: For every $u, v, w$ in $V$ and scalar $c$

i $<u, v> = <v, u>$,

ii $<u + v, w> = <u, w> + <v, w>$,

iii $<cu, v> = c <u, v>$,

iv $<u, u> \geq 0$ and $<u, u> = 0$ if and only if $u = 0$.

A vector space with an inner product is called an inner product space.
Example

Consider $\mathbb{R}^2$, and define the product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 4u_2v_2.$$ 

We saw in an in class assignment that this does define an inner product.
Example

Consider the vector space \( \mathbb{P}_1 \). For polynomials \( p(t) \) and \( q(t) \), show that the product

\[
\langle p, q \rangle = p(0)q(0) + p(1)q(1)
\]
defines an inner product on \( \mathbb{P}_1 \).

Note

\[
\langle \tilde{p}, \tilde{q} \rangle = \tilde{p}(0)\tilde{q}(0) + \tilde{p}(1)\tilde{q}(1) = \tilde{q}(0)p(0) + \tilde{q}(1)p(1) = \langle q, p \rangle
\]

Property holds

For \( \tilde{p}, \tilde{q} \) in \( \mathbb{P}_1 \), \( (\tilde{p} + \tilde{q})(0) = \tilde{p}(0) + \tilde{q}(0) \)

and

\( (\tilde{p} + \tilde{q})(1) = \tilde{p}(1) + \tilde{q}(1) \).
For \( \bar{p}, \bar{q}, \bar{r} \) in \( \bar{P} \),

\[
\langle \bar{p} + \bar{q}, \bar{r} \rangle = (\bar{p} + \bar{q})(0) \bar{r}(0) + (\bar{p} + \bar{q})(1) \bar{r}(1)
\]

\[
= (\bar{p}(0) + \bar{q}(0)) \bar{r}(0) + (\bar{p}(1) + \bar{q}(1)) \bar{r}(1)
\]

\[
= \bar{p}(0) \bar{r}(0) + \bar{q}(0) \bar{r}(0) + \bar{p}(1) \bar{r}(1) + \bar{q}(1) \bar{r}(1)
\]

\[
= \bar{p}(0) \bar{r}(0) + \bar{p}(1) \bar{r}(1) + \bar{q}(0) \bar{r}(0) + \bar{q}(1) \bar{r}(1)
\]

\[
= \langle \bar{p}, \bar{r} \rangle + \langle \bar{q}, \bar{r} \rangle \quad \text{property ii holds}
\]

\[
\langle c\bar{p}, c\bar{q} \rangle = (c\bar{p})(0) c\bar{q}(0) + (c\bar{p})(1) c\bar{q}(1)
\]
\[
\langle \tilde{p}, \tilde{p} \rangle = \tilde{p}(0)^2 + \tilde{p}(1)^2 \\
\geq 0 \quad \text{as a sum of squares.}
\]

Let \( \tilde{p} = p_0 + p_1 t \).

If \( \tilde{p}(0) = 0 \), then \( \tilde{p}(0) = p_0 + p_1 \cdot 0 = p_0 \).

So \( p_0 = 0 \).
If $\hat{p}(1) = 0$ too, then $\hat{\rho}(1) = 0 + \rho_v(1) = \rho_v = 0$

So the only way that $\langle \hat{p}, \hat{p} \rangle = 0$

is if $\hat{p}(t) = 0$

so property iv holds.
Example
Show that the product in the previous example is not an inner product on \( P_2 \) by showing that the last axiom does not hold. In particular, show that there is a nonzero polynomial \( p \) for which \( \langle p, p \rangle = 0 \).

\( \tilde{p} \) in \( P_2 \) has the form \( \tilde{p} = p_0 + p_1 t + p_2 t^2 \)

\( \tilde{p}(0) = 0 \Rightarrow p_0 = 0 \)

\( \tilde{p}(1) = 0 \Rightarrow p_1 + p_2 = 0 \Rightarrow p_1 = -p_2 \)

Any \( \tilde{p} \) that looks like \( \tilde{p} = p_1 (t-t^2) \) would have \( \langle \tilde{p}, \tilde{p} \rangle = 0 \) for this product.
If you consider polynomials \( p \) with \( p(0) = p(1) = 0 \)

There are infinitely many quadratics through the points \((0,0)\) and \((1,0)\).
An Inner Product in $\mathbb{P}_2$

An element in $\mathbb{P}_2$, $p = p_0 + p_1 t + p_2 t^2$, has three defining coefficients $p_0$, $p_1$, and $p_2$. So it is not surprising that evaluation at two points is not sufficient to define an inner product. The following does define an inner product on $\mathbb{P}_2$:

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$
Norm, Distance, and Orthogonality

**Norm:** The norm of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

**A Unit Vector:** is a vector whose norm is 1.

**Distance:** The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.

**Orthogonality:** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

**Orthogonal Projection:** The orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector

$$\hat{\mathbf{v}} = \left( \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \right) \mathbf{u}.$$

**Pythagorean Theorem:** If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$
Example

(a) For the inner product on $\mathbb{P}_1$ in the previous example, find the norm of $p(t) = 1 + t$.

$$
\| \overrightarrow{p} \|^2 = \langle p, p \rangle = p(0)p(0) + p(1)p(1) = 1^2 + 2^2 = 5
$$

$$
\| \overrightarrow{p} \| = \sqrt{5}
$$

(b) Find a unit vector in the direction (i.e. a scalar multiple) of $p(t)$.

Calling in $\hat{u}$

$$
\hat{u} = \frac{1}{\| \overrightarrow{p} \|} \overrightarrow{p} = \frac{1}{\sqrt{5}} (1+t)
$$
Example
(c) Find a polynomial $q(t) = q_0 + q_1 t$ that is orthogonal to $p(t) = 1 + t$.

We need $\langle p, q \rangle = 0$

$0 = p(0) q(0) + p(1) q(1)$

$= 1 q_0 + 2 (q_0 + q_1) = 3q_0 + 2q_1,$

An example would be $q_0 = 2, q_1 = -3$

So $q = 2 - 3t$ is a vector orthogonal

to $p = 1 + t$. 