October 31 Math 3260 sec. 57 Fall 2017

Section 6.4: Gram-Schmidt Orthogonalization

The goal here is to obtain an orthogonal basis for a vector space. The Gram-Schmidt process will allow us to generate an orthogonal basis if we start with an arbitrary basis.

Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1}\left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right)\mathbf{v}_j.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. Moreover, for each $k = 1, \ldots, p$

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for

Col A where $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

We'll start with the columns of A

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\vec{x}_2 \cdot \vec{v}_1 = -6 - 24 - 2 - 4 = -36$$

$$\|\vec{v}_1\|^2 = 1 + 9 + 1 + 1 = 12$$

$$\vec{v}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-36}{12}\right)\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$\vec{x}_3 \cdot \vec{v}_1 = -6 + 9 + 6 - 3 = 6 \qquad \|\vec{v}_1\|^2 = 12$$

$$\vec{x}_3 \cdot \vec{v}_2 = 18 + 3 + 6 + 3 = 30 \qquad \|\vec{v}_2\|^2 = 12$$

$$\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \overset{1/2}{\frac{6}{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \overset{5/2}{\frac{30}{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 15/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} 6-7 \\ 3-4 \\ 6-3 \\ -3+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$\|\vec{v}_3\|^2 = 1+1+9+1 = 12$$

$$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = \sqrt{12}$$

we have orthogonal basis

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

An orthonormal basis is

$$\left\{ \begin{bmatrix} -1/\sqrt{12} \\ \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{12} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{-1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{12} \\ \frac{-1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ \frac{-1}{\sqrt{12}} \end{bmatrix} \right\}$$

Section 6.7 Inner Product Spaces

Definition: An **inner product** on a vector space V is a function which assigns to each pair of vectors **u** and **v** in V a real number denoted by $<\mathbf{u},\mathbf{v}>$ and that satisfies the following four axioms: For every **u**, **v**, **w** in V and scalar c

i $<\mathbf{u},\mathbf{v}>=<\mathbf{v},\mathbf{u}>$,

ii $<\mathbf{u}+\mathbf{v},\mathbf{w}>=<\mathbf{u},\mathbf{w}>+<\mathbf{v},\mathbf{w}>$,

iii $<c\mathbf{u},\mathbf{v}>=c<\mathbf{u},\mathbf{v}>$,

iv $<\mathbf{u},\mathbf{u}>\geq 0$ and $<\mathbf{u},\mathbf{u}>=0$ if and only if $\mathbf{u}=\mathbf{0}$.

A vector space with an inner product is called an **inner product space.**

Example

Consider $\mathbb{R}^2$, and define the product

$$< \mathbf{u}, \mathbf{v} >= 2u_1v_1 + 4u_2v_2.$$

We saw in an in class assignment that this does define an inner product.

Example

Consider the vector space $\mathbb{P}_1$. For polynomials $\mathbf{p}(t)$ and $\mathbf{q}(t)$, show that the product

$$< \mathbf{p}, \mathbf{q} >= p(0)q(0) + p(1)q(1)$$

defines an inner product on $\mathbb{P}_1$.

Note

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}(0)\vec{q}(0) + \vec{p}(1)\vec{q}(1) = \vec{q}(0)\vec{p}(0) + \vec{q}(1)\vec{p}(1) = \langle \vec{q}, \vec{p} \rangle$$

Property i holds

For $\vec{p}, \vec{q}$ in $\mathbb{P}_1$, $(\vec{p}+\vec{q})(0) = \vec{p}(0) + \vec{q}(0)$

and

$$(\vec{p}+\vec{q})(1) = \vec{p}(1) + \vec{q}(1)$$

For $\vec{p}, \vec{q}, \vec{r}$ in $\mathbb{P}_1$

$$\langle \vec{p}+\vec{q}, \vec{r}\rangle = (\vec{p}+\vec{q})(0)\vec{r}(0) + (\vec{p}+\vec{q})(1)\vec{r}(1)$$

$$= (\vec{p}(0)+\vec{q}(0))\vec{r}(0) + (\vec{p}(1)+\vec{q}(1))\vec{r}(1)$$

$$= \vec{p}(0)\vec{r}(0) + \vec{q}(0)\vec{r}(0) + \vec{p}(1)\vec{r}(1) + \vec{q}(1)\vec{r}(1)$$

$$= \vec{p}(0)\vec{r}(0) + \vec{p}(1)\vec{r}(1) + \vec{q}(0)\vec{r}(0) + \vec{q}(1)\vec{r}(1)$$

$$= \langle \vec{p}, \vec{r}\rangle + \langle \vec{q}, \vec{r}\rangle$$ property ii holds

$$\langle c\vec{p}, \vec{q}\rangle = (c\vec{p})(0)\vec{q}(0) + (c\vec{p})(1)\vec{q}(1)$$

$$= c\vec{p}(0)\vec{q}(0) + c\vec{p}(1)\vec{q}(1) = c\left(\vec{p}(0)\vec{q}(0) + \vec{p}(1)\vec{q}(1)\right)$$

$$= c\langle \vec{p}, \vec{q}\rangle.$$

property iii holds.

$$\langle \vec{p}, \vec{p}\rangle = \vec{p}(0)\vec{p}(0) + \vec{p}(1)\vec{p}(1)$$

$$= (\vec{p}(0))^2 + (\vec{p}(1))^2 \geq 0$$

as a sum of squares.

Let $\vec{p} = p_0 + p_1 t$.

If $\vec{p}(0) = 0$, then $\vec{p}(0) = p_0 + p_1 \cdot 0 = p_0$

so $p_0 = 0$.

If $\vec{p}(1) = 0$ too, then $\vec{p}(1) = 0 + p_1 \cdot 1 = p_1 = 0$

So the only way that $\langle \vec{p}, \vec{p} \rangle = 0$

is if $\vec{p}(t) = \vec{0}$

So property iv holds.

Example

Show that the product in the previous example is **not** an inner product on $\mathbb{P}_2$ by showing that the last axiom does not hold. In particular, show that there is a nonzero polynomial **p** for which $<\mathbf{p},\mathbf{p}>= 0$.

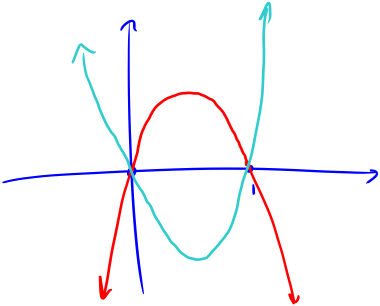
$\vec{p}$ in $\mathbb{P}_2$ has the form $\quad \vec{p} = p_0 + p_1 t + p_2 t^2$

$\vec{p}(0) = 0 \quad \Rightarrow \quad p_0 = 0$

$\vec{p}(1) = 0 \quad \Rightarrow \quad p_1 + p_2 = 0 \quad \Rightarrow \quad p_1 = -p_2$

Any $\vec{p}$ that looks like $\vec{p} = p_1(t - t^2)$ would have $\langle \vec{p}, \vec{p} \rangle = 0$ for this product.

If you consider polynomials w/ $p(0) = p(1) = 0$



There are infinitely many quadratics through the points $(0,0)$ and $(1,0)$.

An Inner Product in $\mathbb{P}_2$

An element in $\mathbb{P}_2$, $\mathbf{p} = p_0 + p_1 t + p_2 t^2$, has three defining coefficients p_0, p_1, and p_2. So it is not surprising that evaluation at two points is not sufficient to define an inner product. The following does define an inner product on $\mathbb{P}_2$:

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$$

Norm, Distance, and Orthogonality

Norm: The norm of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{<\mathbf{v},\mathbf{v}>}$.

A Unit Vector: is a vector whose norm is 1.

Distance: The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}-\mathbf{v}\|$.

Orthogonality: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $<\mathbf{u},\mathbf{v}>= 0$

Orthogonal Projection: The orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector

$$\hat{\mathbf{v}} = \left(\frac{<\mathbf{v},\mathbf{u}>}{<\mathbf{u},\mathbf{u}>}\right)\mathbf{u}.$$

Pythagorean Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Example

(a) For the inner product on $\mathbb{P}_1$ in the previous example, find the norm of $\mathbf{p}(t) = 1 + t$.

$\|\vec{p}\|^2 = \langle \vec{p}, \vec{p} \rangle = p(0)p(0) + p(1)p(1) = 1^2 + 2^2 = 5$

$\|\vec{p}\| = \sqrt{5}$

(b) Find a unit vector *in the direction* (i.e. a scalar multiple) of $\mathbf{p}(t)$.

Calling in $\vec{u}$ $\quad \vec{u} = \dfrac{1}{\|\vec{p}\|}\vec{p} = \dfrac{1}{\sqrt{5}}(1+t)$

Example

(c) Find a polynomial $\mathbf{q}(t) = q_0 + q_1 t$ that is orthogonal to $\mathbf{p}(t) = 1 + t$.

We need $\langle \vec{p}, \vec{q} \rangle = 0$

$$0 = \vec{p}(0)\vec{q}(0) + \vec{p}(1)\vec{q}(1)$$

$$= 1\, q_0 + 2(q_0 + q_1) = 3q_0 + 2q_1$$

An example would be $q_0 = 2$, $q_1 = -3$

So $\vec{q} = 2 - 3t$ is a vector orthogonal to $\vec{p} = 1 + t$.