Section 6.4: Gram-Schmidt Orthogonalization

The goal here is to obtain an orthogonal basis for a vector space. The Gram-Schmidt process will allow us to generate an orthogonal basis if we start with an arbitrary basis.
**Theorem: Gram Schmidt Process**

Let \( \{x_1, \ldots, x_p\} \) be any basis for the nonzero subspace \( W \) of \( \mathbb{R}^n \).

Define the set of vectors \( \{v_1, \ldots, v_p\} \) via

\[
\begin{align*}
  v_1 &= x_1 \\
  v_2 &= x_2 - \left( \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \right) v_1 \\
  v_3 &= x_3 - \left( \frac{x_3 \cdot v_1}{v_1 \cdot v_1} \right) v_1 - \left( \frac{x_3 \cdot v_2}{v_2 \cdot v_2} \right) v_2 \\
  &\vdots \\
  v_p &= x_p - \sum_{j=1}^{p-1} \left( \frac{x_p \cdot v_j}{v_j \cdot v_j} \right) v_j.
\end{align*}
\]

Then \( \{v_1, \ldots, v_p\} \) is an orthogonal basis for \( W \). Moreover, for each \( k = 1, \ldots, p \)

\[
\text{Span}\{v_1, \ldots, v_k\} = \text{Span}\{x_1, \ldots, x_k\}.
\]
Example

Find an orthonormal (that’s *orthonormal* not just orthogonal) basis for $\text{Col } A$ where $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

We'll start with the columns of $A$ as our basis.

- $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$
- $\mathbf{v}_2 = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix}$
- $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$
\[ \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\| \mathbf{v}_1 \|^2} \mathbf{v}_1 \]

\[ \mathbf{x}_2 \cdot \mathbf{v}_1 = -6 \cdot 2 + 2 \cdot 4 - 4 = -36 \]

\[ \| \mathbf{v}_1 \|^2 = 1^2 + 3^2 + 1^2 + 1 = 12 \]

\[ \mathbf{v}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 - 3 \\ -8 + 9 \\ -2 + 3 \\ -4 + 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \]
\[ \vec{V}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{V}_1}{||\vec{V}_1||^2} \vec{V}_1 - \frac{\vec{x}_3 \cdot \vec{V}_2}{||\vec{V}_2||^2} \vec{V}_2 \]

\[ \vec{x}_3 \cdot \vec{V}_1 = -6 + 9 + 6 - 3 = 6 \quad ||\vec{V}_1||^2 = 12 \]

\[ \vec{x}_3 \cdot \vec{V}_2 = 18 + 3 + 6 + 3 = 30 \quad ||\vec{V}_2||^2 = 12 \]

\[ \vec{V}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 - 7 \\ 3 - 4 \\ 6 - 3 \\ -3 + 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \]
An orthogonal basis is

\[ \left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\} \]

\[ \| \mathbf{v}_1 \| = \| \mathbf{v}_2 \| = \| \mathbf{v}_3 \| = \sqrt{12} \]
An orthonormal basis is
\[
\left\{ \begin{bmatrix} \frac{1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix} \right\}.
\]
Section 6.7 Inner Product Spaces

**Definition:** An inner product on a vector space $V$ is a function which assigns to each pair of vectors $u$ and $v$ in $V$ a real number denoted by $\langle u, v \rangle$ and that satisfies the following four axioms: For every $u, v, w$ in $V$ and scalar $c$

i. $\langle u, v \rangle = \langle v, u \rangle$,

ii. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

iii. $\langle cu, v \rangle = c \langle u, v \rangle$,

iv. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

A vector space with an inner product is called an inner product space.
Example

Consider $\mathbb{R}^2$, and define the product

$$\langle u, v \rangle = 2u_1 v_1 + 4u_2 v_2.$$ 

We saw in an in class assignment that this does define an inner product.
Example
Consider the vector space $\mathbb{P}_1$. For polynomials $p(t)$ and $q(t)$, show that the product

$$ \langle p, q \rangle = p(0)q(0) + p(1)q(1) $$

defines an inner product on $\mathbb{P}_1$.

We defined vector addition

$$(p+q)(0) = p(0) + q(0),$$
$$(p+q)(1) = p(1) + q(1).$$
For \( \overline{p}, \overline{q}, \overline{r} \) in \( \mathbb{P} \),

\[
\langle \overline{p} + \overline{q}, \overline{r} \rangle = (\overline{p} + \overline{q})(0) \overline{r}(0) + (\overline{p} + \overline{q})(1) \overline{r}(1)
\]

\[
= (\overline{p}(0) + \overline{q}(0)) \overline{r}(0) + (\overline{p}(1) + \overline{q}(1)) \overline{r}(1)
\]

\[
= \overline{p}(0) \overline{r}(0) + \overline{q}(0) \overline{r}(0) + \overline{p}(1) \overline{r}(1) + \overline{q}(1) \overline{r}(1)
\]

\[
= \overline{p}(0) \overline{r}(0) + \overline{p}(1) \overline{r}(1) + \overline{q}(0) \overline{r}(0) + \overline{q}(1) \overline{r}(1)
\]

\[
= \langle \overline{p}, \overline{r} \rangle + \langle \overline{q}, \overline{r} \rangle.
\]

Property 2 holds.

\[(c \overline{p})(t) = c \overline{p}(t)\]
\[ \langle c \tilde{p}, \tilde{q} \rangle = (C \tilde{p})(0) \tilde{q}(0) + (C \tilde{p})(1) \tilde{q}(1) = c \tilde{p}(0) \tilde{q}(0) + c \tilde{p}(1) \tilde{q}(1) = c \left( \tilde{p}(0) \tilde{q}(0) + \tilde{p}(1) \tilde{q}(1) \right) = c \langle \tilde{p}, \tilde{q} \rangle \]

The property iii holds.

Consider
\[ \langle \tilde{p}, \tilde{p} \rangle = \tilde{p}(0) \tilde{p}(0) + \tilde{p}(1) \tilde{p}(1) = (\tilde{p}(0))^2 + (\tilde{p}(1))^2 \geq 0 \]

This is non-negative. This will be zero.
if and only if \( \dot{\rho}(0) = 0 \) and \( \dot{\rho}(1) = 0 \).

For \( \dot{\rho} \) in \( P_{1} \), \( \dot{\rho} = \rho_{0} + \rho_{1} t \).

\[ \dot{\rho}(0) = \rho_{0} \quad \Rightarrow \quad \rho_{0} = 0 \]

\[ \dot{\rho}(1) = 0 \quad \Rightarrow \quad \rho_{1} = 0 \]

\[ \langle \dot{\rho}, \dot{\rho} \rangle = 0 \quad \text{if and only if} \quad \dot{\rho} = 0. \]

The \( iv^{th} \) property also holds, so this is an inner product on \( P_{1} \).
Example

Show that the product in the previous example is **not** an inner product on $P_2$ by showing that the last axiom does not hold. In particular, show that there is a nonzero polynomial $p$ for which $\langle p, p \rangle = 0$.

$$\langle p, p \rangle = (p(0))^2 + (p(1))^2 > 0$$

In $P_2$, $\bar{p} = p_0 + p_1 t + p_2 t^2$

$p(0) = p_0 \Rightarrow p(0) = 0 \iff p_0 = 0$

$p(1) = p_1 + p_2 \Rightarrow p(1) = 0 \iff p_1 + p_2 = 0$

*This only requires $p_0 = -p_2$*
Any $\vec{p}$ of the form $\vec{p}(t) = p_0 (t-t^2)$ would have $\langle \vec{p}, \vec{p} \rangle = 0$.

Graphically $y = c(x-x^2)$
An Inner Product in $\mathbb{P}_2$

An element in $\mathbb{P}_2$, $p = p_0 + p_1 t + p_2 t^2$, has three defining coefficients $p_0$, $p_1$, and $p_2$. So it is not surprising that evaluation at two points is not sufficient to define an inner product. The following does define an inner product on $\mathbb{P}_2$:

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$
Norm, Distance, and Orthogonality

**Norm:** The norm of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

**A Unit Vector:** is a vector whose norm is 1.

**Distance:** The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.

**Orthogonality:** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

**Orthogonal Projection:** The orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector

$$\hat{\mathbf{v}} = \left( \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \right) \mathbf{u}.$$

**Pythagorean Theorem:** If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$
Example

(a) For the inner product on $\mathbb{P}_1$ in the previous example, find the norm of $p(t) = 1 + t$.

$$\|p\|^2 = \langle p, p \rangle = p(0) \overline{p(0)} + p(1) \overline{p(1)} = 1^2 + 2^2 = 5$$

$$\|p\| = \sqrt{5}$$

(b) Find a unit vector in the direction (i.e. a scalar multiple) of $p(t)$.

Calling it $\hat{u}$, $\hat{u} = \frac{1}{\sqrt{5}} \hat{p} = \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}}t$
Example
(c) Find a polynomial $q(t) = q_0 + q_1 t$ that is orthogonal to $p(t) = 1 + t$.

This requires

$$0 = \langle p, q \rangle = p(0)q(0) + p(1)q(1)$$

$$= 1q_0 + 2(q_0 + q_1) = 3q_0 + 2q_1.$$

An example is when $q_0 = 2$ and $q_1 = -3$

i.e. an example is $\tilde{q} = 2 - 3t$. 
