October 3 MATH 1113 sec. 51 Fall 2018

New Classes of Functions

Polynomial and Rational functions are examples of *Algebraic* functions.

Now we wish to move on to a class of functions called *Transcendental* functions.

We will begin by discussing inverse functions and relations. This will allow us to consider classes of functions called *exponential, logarithmic*, and *trigonometric*.

Section 5.1: Inverse Relations and Inverse Functions

Suppose we have a relation $S = \left\{(1,0),(4,2),\left(\frac{3}{2},\frac{1}{7}\right),(\pi,16),(7,2)\right\}$. If one asks

What is the output if the input is 7?

we can easily answer by referencing the given pairs. Clearly the answer is 2.

One could also pose a similar question:

What is the input if the output is 2?

This question is also easily answered. The answer is 4 or 7.

We can construct another relation from S by interchanging the inputs and outputs. We'll call this the **inverse of** S and denote it as follows

$$S^{-1} = \left\{(0,1),(2,4),\left(\frac{1}{7},\frac{3}{2}\right),(16,\pi),(2,7)\right\}.$$

Inverse Relation

Definition: Let S be a relation with domain D and range R. The inverse relation S^{-1} is the relation having domain R and range D defined by[1]

$$(x, y) \in S^{-1} \quad \text{provided} \quad (y, x) \in S.$$

Recall that a **function** is a type of relation with the property that each domain element is assigned exactly one element from the range—i.e. no distinct pairs in a function can have the same first element.

[1] Recall that $\in$ means "in," so $(x, y) \in S^{-1}$ is read "(x, y) is an element of S^{-1}."

Algebraic Representations

If a relation is defined by an equation, the inverse relation can be written by swapping the variable names. For example, if we consider the relation defined by

$$y = x^2,$$

it's inverse relation would be given by the formula

$$x = y^2.$$

The domains and ranges might be specified or may be infered.

Example

Consider the **functions** $f_1 = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$ and $f_2 = \{(0,0),(1,1),(2,4)\}$.

(a) Write the inverse relations f_1^{-1} and f_2^{-1}.

$$f_1^{-1} = \{(4,-2),\ (1,-1),\ (0,0),\ (1,1),\ (4,2)\}$$

$$f_2^{-1} = \{(0,0),\ (1,1),\ (4,2)\}$$

Example

Consider the **functions** $f_1 = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$ and $f_2 = \{(0,0),(1,1),(2,4)\}$.

(b) Identify the domain and range of each inverse relation.

The domain of f_1^{-1} is $\{4,1,0\}$

(This is the range of f_1)

The range of f_1^{-1} is $\{-2,-1,0,1,2\}$

(This is f_1's domain)

The domain of f_2^{-1} is $\{0,1,4\}$

The range of f_2^{-1} is $\{0,1,2\}$

Question

In the previous example we found the two inverse relations

$$f_1^{-1} = \{(4,-2),(1,-1),(0,0),(1,1),(4,2)\} \quad \text{and} \quad f_2^{-1} = \{(0,0),(1,1),(4,2)\}.$$

Which of the following statements is true?

(a) Neither of these inverse relations are a function.

(b) f_1^{-1} is a function, but f_2^{-1} is not a function.

(c) f_1^{-1} is not a function, but f_2^{-1} is a function.

(d) Since f_1 and f_2 were functions, both inverse relations are also functions.

Inverse Functions

If we look carefully at the function

$$f_1 = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$$

we can see that the inverse relation f_1^{-1} is not going to be a function. At least one second element number appears more than once. This gives us insight into what must be true of a function for its inverse relation to also be a function (called its *inverse function*).

A function will have an inverse **function** is each OUTPUT occurs exactly once! There's a name for this.

One to One

Definition: A function f is **one to one** if different inputs have different outputs. That is, f is one to one provided

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

Equivalently, f is a one to one function provided

$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

Inverse Function

Theorem: If f is a one to one function with domain D and range R, then its inverse f^{-1} is a function with domain R and range D. Moreover, the inverse function is defined by

$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x.$$

↓

"f inverse of x equals y"

Characteristic Compositions:
If f is a one to one function with domain D, range R, and with inverse function f^{-1}, then

- for each x in D, $(f^{-1} \circ f)(x) = x$, and
- for each x in R, $(f \circ f^{-1})(x) = x$.

Example: $f(x) = 3x - 2$

Show that f is one to one.

We'll show that $f(a) = f(b)$ implies $a = b$.

Suppose a and b are in the domain of f and $f(a) = f(b)$. Then

$3a - 2 = 3b - 2$ (add 2 to both sides)

$3a = 3b$ (divide by 3)

$a = b$

So f is one to one.

Example: $f(x) = 3x - 2$

Verify that $f^{-1}(x) = \frac{1}{3}(x + 2)$ by showing that $(f^{-1} \circ f)(x) = x$.

$$(f^{-1} \circ f)(x) = f^{-1}(f(x))$$
$$= f^{-1}(3x-2)$$
$$= \frac{1}{3}((3x-2)+2)$$
$$= \frac{1}{3}(3x-2+2)$$
$$= \frac{1}{3}(3x)$$
$$= x$$