# Oct. 3 Math 1190 sec. 51 Fall 2016

## Section 3.3: Derivatives of Logarithmic Functions

**Recall:** If $a > 0$ and $a \neq 1$, we denote the **base** $a$ **logarithm** of $x$ by

$$\log_a x$$

This is the inverse function of the (one to one) function $y = a^x$. So we can define $\log_a x$ by the statement

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

Our present goal is to use our knowledge of the derivative of an exponential function, along with the chain rule, to come up with a derivative rule for logarithmic functions.

# Properties of Logarithms

We recall several useful properties of logarithms.

Let $a, b, x, y$ be positive real numbers with $a \neq 1$ and $b \neq 1$, and let $r$ be any real number.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r\log_a(x)$
- $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ (*the change of base formula*)

  in particular $\log_a(x) = \frac{\ln x}{\ln a}$
- $\log_a(1) = 0$

# Questions

(1) In the expression $\ln(x)$, what is the base?

(a) 10

(b) 1

(c) $e$

(2) Which of the following expressions is equivalent to

$$\log_2\left(x^3\sqrt{y^2-1}\right)$$

(a) $\log_2(x^3) - \frac{1}{2}\log_2(y^2-1)$

(b) $\frac{3}{2}\log_2(x(y^2-1))$

(c) $3\log_2(x) + \frac{1}{2}\log_2(y^2-1)$

(d) $3\log_2(x) + \frac{1}{2}\log_2(y^2) - \frac{1}{2}\log_2(1)$

# Properties of Logarithms

Additional properties that are useful.

- $f(x) = \log_a(x)$, has domain $(0, \infty)$ and range $(-\infty, \infty)$.
- For $a > 1$, *

$$\lim_{x \to 0^+} \log_a(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} \log_a(x) = \infty$$

- For $0 < a < 1$,

$$\lim_{x \to 0^+} \log_a(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} \log_a(x) = -\infty$$

In advanced mathematics (and in light of the change of base formula), we usually restrict our attention to the natural log.

# Graphs of Logarithms:Logarithms are continuous on $(0, \infty)$.

base smaller than 1

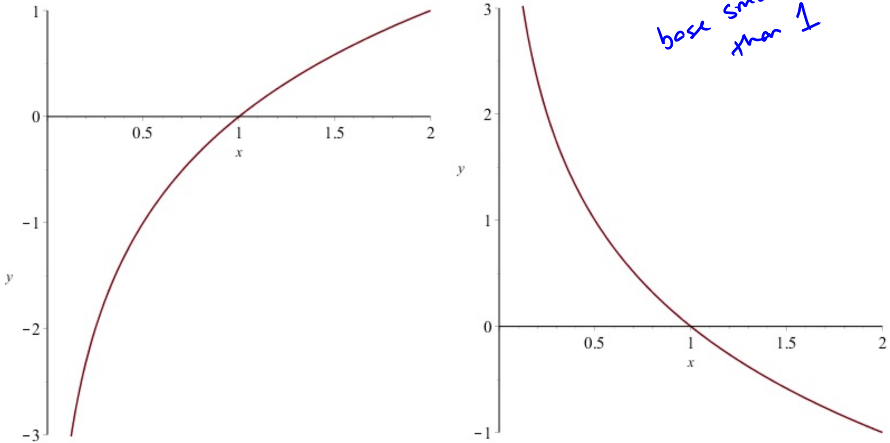



Figure: Plots of functions of the type $f(x) = \log_a(x)$. The value of $a > 1$ on the left, and $0 < a < 1$ on the right.

# Examples

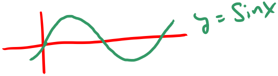
Evaluate each limit.

(a) $\lim_{x \to 0^+} \ln(\sin(x)) = -\infty$

Note $e > 1$

$\lim_{x \to 0^+} \ln(x) = -\infty$

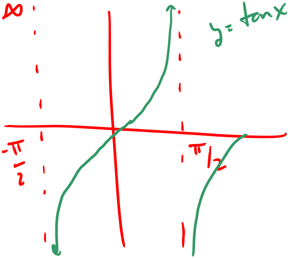
as $x \to 0^+$, $\sin x \to 0^+$




(b) $\lim_{x \to \frac{\pi}{2}^-} \ln(\tan(x)) = \infty$

$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$

$\lim_{x \to \infty} \ln x = \infty$



# Questions

$\lim_{x \to 0^+} \ln x = -\infty$

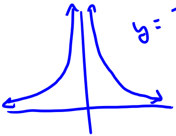
$\lim_{x \to \infty} \ln x = \infty$

Evaluate the limit $\lim_{x \to \infty} \ln\left(\frac{1}{x^2}\right)$

$\lim_{x \to \infty} \frac{1}{x^2} = 0$

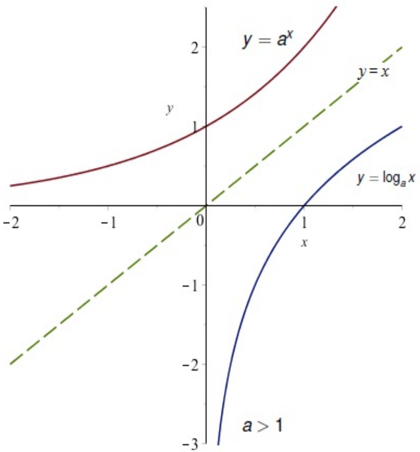
and it's going to zero through positive numbers.

(a) $-\infty$
(b) 0
(c) $\infty$
(d) The limit doesn't exist.




as $x \to \infty$, $\frac{1}{x^2} \to 0^+$

# Logarithms are Differentiable on Their Domain



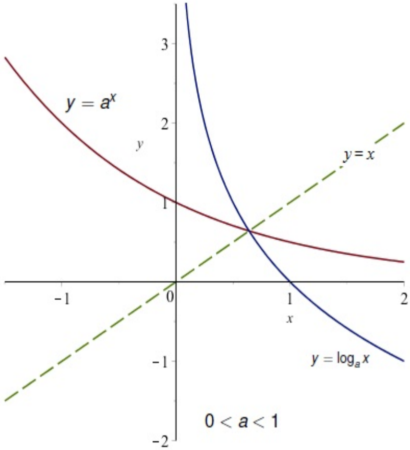



Figure: Recall $f(x) = a^x$ is differentiable on $(-\infty, \infty)$. The graph of $\log_a(x)$ is a reflection of the graph of $a^x$ in the line $y = x$. So $f(x) = \log_a(x)$ is differentiable on $(0, \infty)$.

# The Derivative of $y = \log_a(x)$

To find a derivative rule for $y = \log_a(x)$, we use the chain rule (i.e. implicit differentiation).

Let $y = \log_a(x)$, then $x = a^y$.

$$\frac{d}{dx} x = \frac{d}{dx} a^y$$

$$1 = a^y \ln a \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{a^y \ln a}$$

but $a^y = x$, so

$$\boxed{\frac{d}{dx} \log_a x = \frac{1}{x \ln a}}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

Examples: Evaluate each derivative.

(a) $\frac{d}{dx}\log_3(x) = \frac{1}{x \ln 3}$ , here $a = 3$

(b) $\frac{d}{d\theta}\log_{\frac{1}{2}}(\theta) = \frac{1}{\theta \ln(\frac{1}{2})}$ , here $a = \frac{1}{2}$

# Question

$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

**True or False** The derivative of the natural log

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

True. $a = e$ and $\ln e = 1$

# The function $\ln|x|$

Recall that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. Use this to derive the rule

$$\frac{d}{dx} \ln|x| = \frac{1}{x}.$$

If $x > 0$, then $|x| = x$. So for $x > 0$

$\ln|x| = \ln x$ and

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln x = \frac{1}{x}$$

If $x < 0$, $|x| = -x$, So for $x<0$

$$\ln|x| = \ln(-x), \text{ and}$$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{(-x)} \cdot \frac{d}{dx}(-x)$$

Chain rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$$

$$= \frac{-1}{-x} = \frac{1}{x}$$

Combined, we have $\frac{d}{dx}\ln|x| = \frac{1}{x}$.

# Differentiating Functions Involving Logs

We can combine our new rule with our existing derivative rules.

**Chain Rule:** Let $u$ be a differentiable function. Then

$$\frac{d}{dx}\log_a(u) = \frac{1}{u\ln(a)}\frac{du}{dx} = \frac{u'(x)}{u(x)\ln(a)}.$$

In particular

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx} = \frac{u'(x)}{u(x)}.$$

# Examples

Evaluate each derivative.

(a) $\dfrac{d}{dx} \ln|\tan x|$

$= \dfrac{\sec^2 x}{\tan x}$

absolute value bars don't affect the derivative rule for the log.

here

$u = \tan x$

and

$u' = \sec^2 x$

# Example

(b) $\frac{d}{dt}\sqrt{\log_3(t)} = \frac{d}{dt}\left(\log_3(t)\right)^{1/2}$

outside is
$\frac{1}{2}$ power

inside is $\log_3 t$

$= \frac{1}{2}\left(\log_3(t)\right)^{\frac{1}{2}-1} \cdot \frac{1}{t \ln 3}$

$= \frac{1}{2}\left(\log_3(t)\right)^{-1/2} \cdot \frac{1}{t \ln 3} = \frac{1}{2t \ln 3 \sqrt{\log_3(t)}}$

# Example

Determine $\frac{dy}{dx}$ if $x \ln y + y \ln x = 10.$

$$\frac{d}{dx}\left(x \ln y + y \ln x\right) = \frac{d}{dx} 10$$

products

$$\left(\frac{d}{dx} x\right) \ln y + x\left(\frac{d}{dx} \ln y\right) + \left(\frac{d}{dx} y\right) \ln x + y\left(\frac{d}{dx} \ln x\right) = 0$$

Clear fractions

$$1 \cdot \ln y + x \cdot \frac{1}{y} \frac{dy}{dx} + \frac{dy}{dx} \ln x + y \cdot \frac{1}{x} = 0$$

$$xy\left(\ln y + \frac{x}{y} \frac{dy}{dx} + \ln x \frac{dy}{dx} + \frac{y}{x}\right) = (0)\, xy$$

$$xy \ln y + x^2 \frac{dy}{dx} + xy \ln x \frac{dy}{dx} + y^2 = 0$$

$$\left(x^2 + xy \ln x\right) \frac{dy}{dx} = -xy \ln y - y^2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{-xy \ln y - y^2}{x^2 + xy \ln x}}$$

# Questions

Find $y'$ if $y = x\,(\ln x)^2$.

product rule w/ chain rule

(a) $y' = \frac{2\ln x}{x}$

$$y' = 1 \cdot (\ln x)^2 + x\left[2(\ln x)^1 \cdot \frac{1}{x}\right]$$

(b) $y' = 2\ln x + 2$

$$= (\ln x)^2 + 2\ln x$$

(c) $y' = (\ln x)^2 + 2\ln x$

(d) $y' = \ln(x^2) + 2$

# Questions

Use implicit differentiation to find $\frac{dy}{dx}$ if $\quad y^2 \ln x = x + y$.

(a) $\quad \frac{dy}{dx} = \frac{x - y^2}{2xy \ln x - x}$

(b) $\quad \frac{dy}{dx} = \frac{1}{2y \ln x - 1}$

(c) $\quad \frac{dy}{dx} = y^2 \ln x - 1$

(c) $\quad \frac{dy}{dx} = \frac{x}{2y - x}$

$$\frac{d}{dx} y^2 \ln x = \frac{d}{dx}(x+y)$$

$$2y\frac{dy}{dx}\ln x + y^2 \cdot \frac{1}{x} = 1 + \frac{dy}{dx}$$

$$\frac{x}{1}\left(2y\ln x \frac{dy}{dx} + \frac{y^2}{x}\right) = \left(1 + \frac{dy}{dx}\right)\frac{x}{1}$$

$$2xy\ln x \frac{dy}{dx} + y^2 = x + x\frac{dy}{dx}$$