Oct. 3 Math 1190 sec. 52 Fall 2016

Section 3.3: Derivatives of Logarithmic Functions

Recall: If $a > 0$ and $a \neq 1$, we denote the **base** a **logarithm** of x by

$$\log_a x$$

This is the inverse function of the (one to one) function $y = a^x$. So we can define $\log_a x$ by the statement

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

Our present goal is to use our knowledge of the derivative of an exponential function, along with the chain rule, to come up with a derivative rule for logarithmic functions.

Properties of Logarithms

We recall several useful properties of logarithms.

Let a, b, x, y be positive real numbers with $a \neq 1$ and $b \neq 1$, and let r be any real number.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r\log_a(x)$
- $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ (*the change of base formula*)
- $\log_a(1) = 0$

for example

$\log_a x = \frac{\ln x}{\ln a}$

Questions

(1) In the expression $\ln(x)$, what is the base?

(a) 10

(b) 1

(c) e

(2) Which of the following expressions is equivalent to

$$\log_2\left(x^3\sqrt{y^2-1}\right)$$

(a) $\log_2(x^3) - \frac{1}{2}\log_2(y^2-1)$

(b) $\frac{3}{2}\log_2(x(y^2-1))$

(c) $3\log_2(x) + \frac{1}{2}\log_2(y^2-1)$

(d) $3\log_2(x) + \frac{1}{2}\log_2(y^2) - \frac{1}{2}\log_2(1)$

Properties of Logarithms

Additional properties that are useful.

- $f(x) = \log_a(x)$, has domain $(0, \infty)$ and range $(-\infty, \infty)$.
- For $a > 1$, *

$$\lim_{x \to 0^+} \log_a(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} \log_a(x) = \infty$$

- For $0 < a < 1$,

$$\lim_{x \to 0^+} \log_a(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} \log_a(x) = -\infty$$

In advanced mathematics (and in light of the change of base formula), we usually restrict our attention to the natural log.

Graphs of Logarithms:Logarithms are continuous on $(0, \infty)$.

base smaller than 1

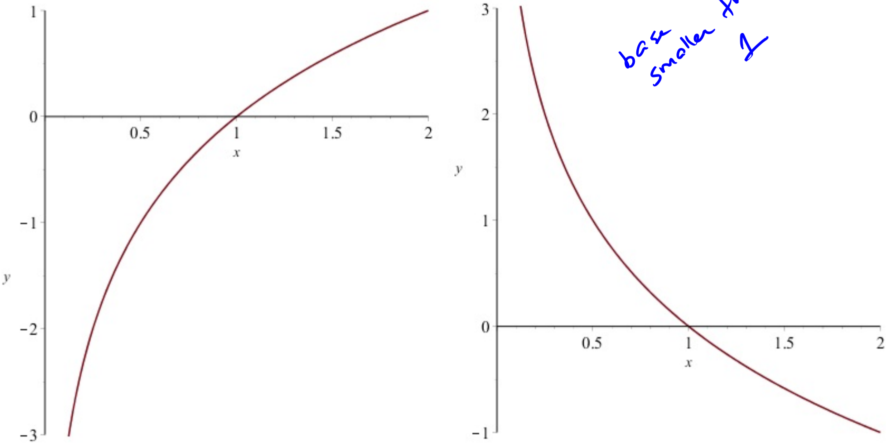


Figure: Plots of functions of the type $f(x) = \log_a(x)$. The value of $a > 1$ on the left, and $0 < a < 1$ on the right.

Examples

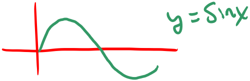
Evaluate each limit.

(a) $\lim_{x \to 0^+} \ln(\sin(x))$ $= -\infty$

(b) $\lim_{x \to \frac{\pi}{2}^-} \ln(\tan(x))$ $= \infty$

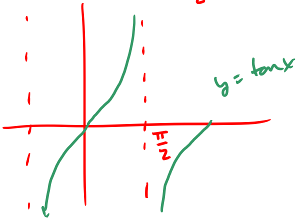
$\lim_{x \to 0^+} \ln x = -\infty$

as $x \to 0^+$, $\sin x \to 0^+$



$\lim_{x \to \infty} \ln x = \infty$

$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$



Questions

$$\lim_{x\to 0^+} \ln x = -\infty$$

$$\lim_{x\to \infty} \ln x = \infty$$

Evaluate the limit $\lim_{x\to\infty} \ln\left(\frac{1}{x^2}\right)$

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

and $\frac{1}{x^2} \to 0^+$

as $x \to \infty$

(a) $-\infty$

(b) 0

(c) ∞

(d) The limit doesn't exist.

Logarithms are Differentiable on Their Domain

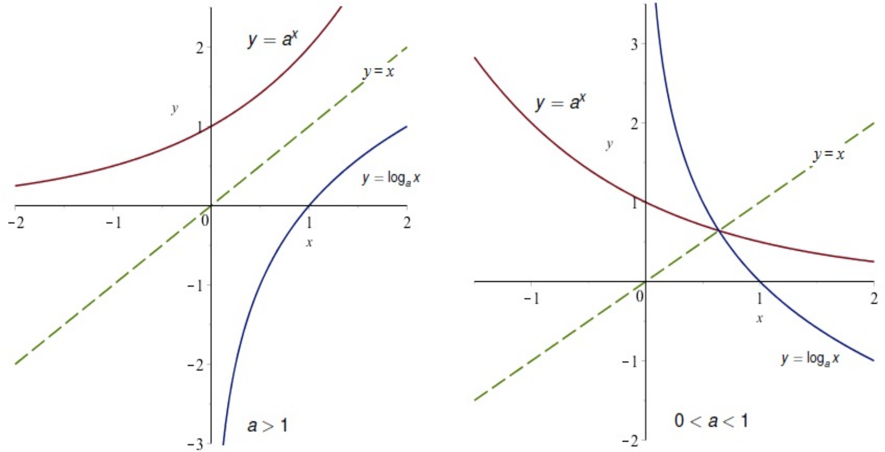


Figure: Recall $f(x) = a^x$ is differentiable on $(-\infty, \infty)$. The graph of $\log_a(x)$ is a reflection of the graph of a^x in the line $y = x$. So $f(x) = \log_a(x)$ is differentiable on $(0, \infty)$.

The Derivative of $y = \log_a(x)$

To find a derivative rule for $y = \log_a(x)$, we use the chain rule (i.e. implicit differentiation).

Let $y = \log_a(x)$, then $x = a^y$.

$$\frac{d}{dx} x = \frac{d}{dx} a^y$$

$$1 = a^y \ln a \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{a^y \ln a}$$

use $x = a^y$

$$\frac{dy}{dx} = \frac{1}{x \ln a} \quad \Rightarrow \quad \boxed{\frac{d}{dx} \log_a x = \frac{1}{x \ln a}}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

Examples: Evaluate each derivative.

(a) $\frac{d}{dx}\log_3(x) = \frac{1}{x\ln 3}$, here $a = 3$

(b) $\frac{d}{d\theta}\log_{\frac{1}{2}}(\theta) = \frac{1}{\theta\ln\left(\frac{1}{2}\right)}$, here $a = \frac{1}{2}$

Question

$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

True or False The derivative of the natural log

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

True. $a = e$ and $\ln e = 1$

The function $\ln|x|$

Recall that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. Use this to derive the rule

$$\frac{d}{dx} \ln|x| = \frac{1}{x}.$$

If $x > 0$, then $|x| = x$. So

$\ln|x| = \ln x$ if $x > 0$, and

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln x = \frac{1}{x}.$$

If $x < 0$, then $|x| = -x$. So for $x < 0$

$\ln|x| = \ln(-x)$. Then by the chain rule

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

So $\frac{d}{dx} \ln|x| = \frac{1}{x}$ for all real $x \neq 0$.

Differentiating Functions Involving Logs

We can combine our new rule with our existing derivative rules.

Chain Rule: Let u be a differentiable function. Then

$$\frac{d}{dx}\log_a(u) = \frac{1}{u\ \ln(a)}\frac{du}{dx} = \frac{u'(x)}{u(x)\ \ln(a)}.$$

In particular

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx} = \frac{u'(x)}{u(x)}.$$

Examples

Evaluate each derivative.

(a) $\frac{d}{dx} \ln|\tan x|$

$= \frac{\sec^2 x}{\tan x}$

here

$u = \tan x$ and

$u'(x) = \sec^2 x$

Example

outside $\sqrt{u}$

inside $\log_3(t)$

(b) $\dfrac{d}{dt}\sqrt{\log_3(t)}$ $= \dfrac{d}{dt}\left(\log_3(t)\right)^{1/2}$

$= \frac{1}{2}\left(\log_3(t)\right)^{\frac{1}{2}-1} \cdot \dfrac{1}{t\ln 3}$

$= \frac{1}{2}\left(\log_3(t)\right)^{-1/2} \cdot \dfrac{1}{t\ln 3}$

$= \dfrac{1}{2t\ln 3\sqrt{\log_3(t)}}$

Example

Determine $\frac{dy}{dx}$ if $\quad x \ln y + y \ln x = 10.$

$$\frac{d}{dx}\left(x \ln y + y \ln x\right) = \frac{d}{dx}(10)$$

products

$$\left(\frac{d}{dx}x\right)\ln y + x\left(\frac{d}{dx}\ln y\right) + \left(\frac{d}{dx}y\right)\ln x + y\left(\frac{d}{dx}\ln x\right) = 0$$

$$1 \cdot \ln y + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \frac{dy}{dx}\ln x + y \cdot \frac{1}{x} = 0$$

Clearing fractions

$$\ln y + \frac{x}{y}\frac{dy}{dx} + \ln x \frac{dy}{dx} + \frac{y}{x} = 0$$

$$xy\left(\ln y + \frac{x}{y}\frac{dy}{dx} + \ln x \frac{dy}{dx} + \frac{y}{x}\right) = 0 \cdot (xy)$$

$$xy\ln y + x^2\frac{dy}{dx} + xy\ln x\frac{dy}{dx} + y^2 = 0$$

$$(x^2 + xy\ln x)\frac{dy}{dx} = -y^2 - xy\ln y$$

$$\boxed{\frac{dy}{dx} = \frac{-y^2 - xy\ln y}{x^2 + xy\ln x}}$$

Questions

Find y' if $y = x\,(\ln x)^2$.

(a) $y' = \frac{2\ln x}{x}$

(b) $y' = 2\ln x + 2$

(c) $y' = (\ln x)^2 + 2\ln x$

(d) $y' = \ln(x^2) + 2$

$$y' = 1 \cdot (\ln x)^2 + x\left(2(\ln x) \cdot \frac{1}{x}\right)$$

$$= (\ln x)^2 + 2\ln x$$

Questions

Use implicit differentiation to find $\frac{dy}{dx}$ if $\quad y^2 \ln x = x + y$.

(a) $\displaystyle \frac{dy}{dx} = \frac{x - y^2}{2xy \ln x - x}$

(b) $\displaystyle \frac{dy}{dx} = \frac{1}{2y \ln x - 1}$

(c) $\displaystyle \frac{dy}{dx} = y^2 \ln x - 1$

(c) $\displaystyle \frac{dy}{dx} = \frac{x}{2y - x}$

$$\frac{d}{dx}\left(y^2 \ln x\right) = \frac{d}{dx}(x+y)$$

$$2y \frac{dy}{dx} \ln x + y^2 \cdot \frac{1}{x} = 1 + \frac{dy}{dx}$$

$$x\left(2y \ln x \frac{dy}{dx} + \frac{y^2}{x}\right) = x\left(1 + \frac{dy}{dx}\right)$$

$$2xy \ln x \frac{dy}{dx} + y^2 = x + x\frac{dy}{dy}$$

$$\vdots$$

Using Properties of Logs

Properties of logarithms can be used to simplify expressions characterized by products, quotients and powers.

Illustrative Example: Evaluate $$\frac{d}{dx}\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right)$$

Note

$$\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \ln\left(x^2\cos(2x)\right) - \ln\sqrt[3]{x^2+x}$$

$$= \ln x^2 + \ln\cos(2x) - \ln\left(x^2+x\right)^{\frac{1}{3}}$$

$$= 2\ln x + \ln\cos(2x) - \frac{1}{3}\ln\left(x^2+x\right)$$

$$\frac{d}{dx}\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \frac{d}{dx}\left(2\ln x + \ln(\cos(2x)) - \frac{1}{3}\ln(x^2+x)\right)$$

Recall

$$\frac{d}{dx}\ln u = \frac{u'(x)}{u(x)}$$

$$= 2\cdot\frac{1}{x} + \frac{-\sin(2x)\cdot 2}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

$$= \frac{2}{x} - \frac{2\sin(2x)}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$