Oct 3 Math 2306 sec. 53 Fall 2018

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, like e^{mx} m- constant
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

constant coefficient

polynomial

We'll assume that since $g(x) = 8x+1$ is a 1^{st} degree polynomial, y_p is also a 1^{st} degree polynomial.

Let $y_p = Ax + B$ for some numbers A + B.

We try to find A, B to fit the ODE.

Substitute:

We require
$$y_p'' - 4y_p' + 4y_p = 8x+1$$

$y_p = Ax+B$

$y_p' = A$

$y_p'' = 0$

$$0 - 4(A) + 4(Ax+B) = 8x+1$$
$$-4A + 4Ax + 4B = 8x+1$$
$$4Ax + (-4A+4B) = 8x+1$$

Match coefficients

$$\left.\begin{aligned} 4A \qquad &= 8 \\ -4A + 4B &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &A = 2 \\ &B = \frac{1}{4}(1+4A) = \frac{1}{4}(1+8) = \frac{9}{4} \end{aligned}$$

So the particular solution is

$$y_p = 2x + \frac{9}{4}$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

We'll assume that $y_p = Ae^{-3x}$

Substitute:

$$y_p' = -3Ae^{-3x}$$

$$y_p'' = 9Ae^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$e^{-3x}(9A + 12A + 4A) = 6e^{-3x}$$

$$25A e^{-3x} = 6e^{-3x}$$

Matching gives

$$25A = 6$$

$$A = \frac{6}{25}$$

Our particular solution is

$$y_p = \frac{6}{25} e^{-3x}.$$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

$g(x) = 16x^2$ is a constant times x^2.

(it is also a quadratic function)

Let's suppose that $y_p = Ax^2$. Substitute

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Matching gives $\left.\begin{array}{r} 4A = 16 \\ -8A = 0 \\ 2A = 0 \end{array}\right\} \Rightarrow$ $A = 4$ and $A = 0$

Not possible!

We didn't allow for x or constant terms. We should think of $g(x) = 16x^2$ as a 2^{nd} degree polynomial. Let $y_p = Ax^2 + Bx + C$.

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Substitute

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Matching:

$$\left.\begin{aligned} 4A \qquad\qquad\qquad &= 16 \\ -8A + 4B \qquad &= 0 \\ 2A - 4B + 4C &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &A = 4 \\ &B = \tfrac{1}{4}(8A) = 2A = 8 \\ &C = \tfrac{1}{4}(4B - 2A) \\ &\quad = \tfrac{1}{4}(32 - 8) = \tfrac{24}{4} = 6 \end{aligned}$$

$$\boxed{\text{So } y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

If $y_p = A\sin(2x) + B\cos(2x)$, then we get

$$\begin{aligned} y_p'' - y_p' &= -4A\sin(2x) - 4B\cos(2x) - 2A\cos(2x) + 2B\sin(2x) \\ &= 20\sin(2x) + 0\cos(2x). \end{aligned}$$

We can match coefficients of $\sin(2x)$ and $\cos(2x)$ giving a **solvable** set of equations

$$-4A + 2B = 20 \quad \text{and} \quad -2A - 4B = 0 \quad \implies \quad A = -4, \quad B = 2.$$

The particular solution is

$$y_p = -4\sin(2x) + 2\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant)

Degree 0 polynomial

$y_p = A$

(b) $g(x) = x - 7$

Degree 1 poly.

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$

Degree 2 poly.

$y_p = Ax^2 + Bx + C$

(d) $g(x) = 3x^3 - 5$

Degree 3 polynomial

$y_p = Ax^3 + Bx^2 + Cx + D$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$

Product of Degree 1 poly and e^{3x}

$$y_p = (Ax+B)e^{3x}$$

(f) $g(x) = \cos(7x)$

Sum of Cosine + Sine $7x$

$$y_p = A\cos(7x) + B\sin(7x)$$