# October 3 Math 3260 sec. 57 Fall 2017

## Section 4.2: Null & Column Spaces, Linear Transformations

**Definition:** Let $A$ be an $m \times n$ matrix. The **null space** of $A$, denoted by Nul $A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

**Theorem:** For $m \times n$ matrix $A$, Nul $A$ is a subspace of $\mathbb{R}^n$.

We saw that we can describe Nul $A$ explicitly by finding a spanning set, and that this can be determined from the rref of $A$.

# Column Space

**Definition:** The **column space** of an $m \times n$ matrix $A$, denoted Col $A$, is the set of all linear combinations of the columns of $A$. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\mathrm{Col} A = \mathrm{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}.$$

Note that this corresponds to the set of solutions **b** of linear equations of the form $A\mathbf{x} = \mathbf{b}$! That is

$$\mathrm{Col} A = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

# Theorem

The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.

The columns $\vec{a}_i$ are in $\mathbb{R}^m$.

This was hinted at in the worksheet on Sept. 14 and problem #5 on exam 1.

**Corollary:** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

# Example

Find a matrix $A$ such that $W = \text{Col}\, A$ where

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

We'll look for a spanning set which provides columns for $A$. Consider $\vec{u}$ in $W$

$$\vec{u} = \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} = a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

So $\vec{u}$ is in $\text{Span}\left\{\begin{bmatrix}6\\1\\-7\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$

We can take

$$A = \begin{bmatrix}6 & -1\\1 & 1\\-7 & 0\end{bmatrix}.$$

# Example

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

(a) If Col $A$ is a subspace of $\mathbb{R}^k$, what is $k$?

$k = 3$, the columns are in $\mathbb{R}^3$

(b) If Nul $A$ is a subspace of $\mathbb{R}^k$, what is $k$?

$k = 4$ $\quad A\vec{x}$ is defined for $\vec{x}$ in $\mathbb{R}^4$

# Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

(c) Is **u** in Nul $A$? Could **u** be in Col $A$?

$\vec{u}$ can't be in Col A since Col A is a subset of $\mathbb{R}^3$ and $\vec{u}$ is in $\mathbb{R}^4$. Compute $A\vec{u} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0}$.

$\vec{u}$ is not in Nul A.

# Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(c) Is **v** in Col $A$? Could **v** be in Nul $A$?

$\vec{v}$ could not be in Nul $A$. In fact $A\vec{v}$ isn't defined.

If $\vec{v}$ is in Col $A$, then there is $\vec{x}$ in $\mathbb{R}^4$ such that

$$A\vec{x} = \vec{v}.$$

We can row reduce the augmented matrix

$[A \ \vec{v}]$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \xrightarrow{\text{rref}}$$

$$\begin{bmatrix} 1 & 0 & 9 & 0 & 5 - 30/17 \\ 0 & 1 & -5 & 0 & \\ 0 & 0 & 0 & 1 & 1/17 \end{bmatrix}$$

$A\vec{x} = \vec{v}$ is consistent

not a pivot column

$\vec{v}$ is in Col $A$.

# Linear Transformation

**Definition:** Let $V$ and $W$ be vector spaces. A linear transformation $T : V \longrightarrow W$ is a rule that assigns to each vector $\mathbf{x}$ in $V$ a unique vector $T(\mathbf{x})$ in $W$ such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in $V$, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in $V$ and scalar $c$.

# Example

Let $C^1(\mathbb{R})$ denote the set of all real valued functions that are differentiable and $C^0(\mathbb{R})$ the set of all continuous real valued functions. Note that differentiation is a linear transformation. That is

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

satisfies the two conditions in the previous definition.

We know that $\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x) + g'(x)$

i.e. $D(f+g) = f'+g' = D(f) + D(g)$

And $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$ i.e. $D(cf) = cf' = cD(f)$

Characterize the subset of $C^1(\mathbb{R})$ such that $Df = 0$. ← this is the zero function

This is the subset of constant functions $f(x) = k$.

# Range and Kernel

**Definition:** The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in $W$ of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in $V$. (The set of all images of elements of $V$.) Analog of a column space.

**Definition:** The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in $V$ such that $T(\mathbf{x}) = \mathbf{0}$. (The analog of the null space of a matrix.)

**Theorem:** Given linear transformation $T : V \longrightarrow W$, the range of $T$ is a subspace of $W$ and the kernel of $T$ is a subspace of $V$.

# Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function $y$ must satisfy if $y$ is in the kernel of $T$.

If $y$ is in the kernel of $T$, then $T(y) = 0$.

the zero function.

This requires

$$\frac{dy}{dx} + \alpha y = 0 .$$

# Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(b) Show that for any scalar $c$, $y = ce^{-\alpha x}$ is in the kernel of $T$.

If $y = ce^{-\alpha x}$, then $\frac{dy}{dx} = -\alpha c e^{-\alpha x}$. For such $y$

$T(y) = \frac{dy}{dx} + \alpha y = -\alpha c e^{-\alpha x} + \alpha c e^{-\alpha x} = 0$

so such $y$ is in the kernel of $T$.

# Section 4.3: Linearly Independent Sets and Bases

**Definition:** A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space $V$ is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.
The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights $c_i$ is nonzero). If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation.**

**Theorem:** The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ for $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

# Example

Determine if the set is linearly dependent or independent in $\mathbb{P}_2$.

(a) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = 2t$, $\mathbf{p}_3 = t - 3$.

Consider $c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}$

Note $\vec{p}_3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1$ Hence $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is linearly dependent. A linear dependence relation is

$$3\vec{p}_1 - \frac{1}{2}\vec{p}_2 + \vec{p}_3 = \vec{0}.$$

(b) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

Again, consider $c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}$

$$2c_1 + c_2 t - c_3 t^2 = 0 + 0t + 0t^2$$

This must hold for all real (or complex) $t$.

So $c_1 = c_2 = c_3 = 0$.

This set is linearly independent.

# Example

Show that every vector $\mathbf{p} = p_0 + p_1 t + p_2 t^2$ in $\mathbb{P}_2$ can be written as a linear combination of $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$[1] where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

If we take $a_0 = \frac{1}{2}p_0$ , $a_1 = p_1$ and $a_2 = -p_2$

then

$$a_0\vec{p}_1 + a_1\vec{p}_2 + a_2\vec{p}_3 = \frac{1}{2}p_0(2) + p_1(t) + (-p_2)(-t^2)$$

$$= p_0 + p_1 t + p_2 t^2$$

[1] i.e. this set *spans* $\mathbb{P}_2$

# Definition (Basis)

**Definition:** Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in $V$ is a **basis** of $H$ provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in $H$ is contained in the basis, and none of this information is repeated.

# Example

If $A$ is an invertible $n \times n$ matrix, then we know[2] that (1) the columns are linearly independent, and (2) the columns span $\mathbb{R}^n$. Use this to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

We can form the matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$.

One approach is to use the determinant

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$$

[2] from our large theorem on invertible matrices from section 2.3

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= 3\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} + 0\begin{vmatrix} & \\ & \end{vmatrix} - 6\begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3(5-7) - 6(-4+2) = -6 + 12 = 6$$

$\det(A) \neq 0$ is equivalent to the columns being ① linearly independent and ② spanning $\mathbb{R}^3$.

Hence $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

## Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$

# Other Vector Spaces

Show that $\{1, t, t^2, t^3\}$ is a basis for $\mathbb{P}_3$[3].

① Note $c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3 = 0 + 0t + 0t^2 + 0t^3$

$\Rightarrow \quad c_1 = c_2 = c_3 = c_4 = 0$

The set is linearly independent.

② For $\vec{p} = p_0 + p_1 t + p_2 t^2 + p_3 t^3$ in $\mathbb{P}_3$, we have

$\vec{p}$ as a linear combination of the elements

with corresponding coefficients $p_i$.

[3] The set $\{1, t, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$

# Other Vector Spaces

Show that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M^{2\times 2}$.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $M^{2\times 2}$.

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The set spans $M^{2\times 2}$.

If $A = 0$ then

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

requires $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, i.e. $c_1 = c_2 = c_3 = c_4 = 0$.

So the set is linearly independent.

# A Spanning Set Theorem

**Example:** Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space $V$, and suppose that
(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.
Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

For $\vec{u}$ in $H$, $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2)$

$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2$

$= d_1\vec{v}_1 + d_2\vec{v}_2$

when $d_1 = c_1 + c_3$ and $d_2 = c_2 - 2c_3$.

So $\vec{u}$ is in Span$\{\vec{v}_1, \vec{v}_2\}$.

# Theorem:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space $V$ and $H =$Span($S$).

(a.) If one of the vectors in $S$, say $\mathbf{v}_k$ is a linear combination of the other vectors in $S$, then the subset of $S$ obtained by eliminating $\mathbf{v}_k$ still spans $H$.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of $S$ is a basis for $H$.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.