October 3 Math 3260 sec. 58 Fall 2017

Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

Theorem: For $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.

We saw that we can describe Nul A explicitly by finding a spanning set, and that this can be determined from the rref of A.

Column Space

Definition: The **column space** of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\text{Col}A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Note that this corresponds to the set of solutions $\mathbf{b}$ of linear equations of the form $A\mathbf{x} = \mathbf{b}$! That is

$$\text{Col}A = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

That Col A is a subset of $\mathbb{R}^m$ follows from the columns being in $\mathbb{R}^m$.

This was also hinted at in the worksheet from Sept. 14 and problem # 5 from exam 1.

Corollary: Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Example

Find a matrix A such that $W = \text{Col}\, A$ where

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

For $\vec{u}$ in W, for some real numbers a and b

$$\vec{u} = \begin{bmatrix} 6a - b \\ a+b \\ -7a \end{bmatrix} = a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

so $\vec{u}$ is in $\text{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

We can take

$$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

(a) If Col A is a subspace of $\mathbb{R}^k$, what is k?

$k = 3$

(b) If Nul A is a subspace of $\mathbb{R}^k$, what is k? $k = 4$

$A\vec{x}$ requires $\vec{x}$ in $\mathbb{R}^4$ to be defined.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

(c) Is **u** in Nul A? Could **u** be in Col A?

$\vec{u}$ cannot be in Col A since $\vec{u}$ is in $\mathbb{R}^4$ and Col A is a subspace of $\mathbb{R}^3$. We can compute $A\vec{u}$.

$A\vec{u} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0}$. $\vec{u}$ is not in Nul A.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(c) Is **v** in Col A? Could **v** be in Nul A?

$\vec{v}$ can not be in Nul A. In fact $A\vec{v}$ isn't defined.

To determine if $\vec{v}$ is in Col A, we can determine if the system $A\vec{x}=\vec{v}$ is consistent. We can use an augmented matrix $[A \ \vec{v}]$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \xrightarrow{\text{rref}}$$

$$\begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & -30/17 \\ 0 & 0 & 0 & 1 & 1/17 \end{bmatrix} \Rightarrow A\vec{x} = \vec{v} \text{ is consistent}$$

not a pivot column

$\vec{v}$ is in Col A.

Linear Transformation

Definition: Let V and W be vector spaces. A linear transformation $T : V \longrightarrow W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and scalar c.

Example

Let $C^1(\mathbb{R})$ denote the set of all real valued functions that are differentiable and $C^0(\mathbb{R})$ the set of all continuous real valued functions. Note that differentiation is a linear transformation. That is

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

satisfies the two conditions in the previous definition.

For f, g in $C^1(\mathbb{R})$, recall $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$

That is $D(f+g) = f'+g' = D(f)+D(g)$.

Similarly $\frac{d}{dx}(cf(x)) = cf'(x)$, hence $D(cf) = cf' = cD(f)$.

Characterize the subset of $C^1(\mathbb{R})$ such that $Df = 0$. ← the zero function in $C^0(\mathbb{R})$

This is the set of constant functions

$f(x) = k$ for all x.

Range and Kernel

Definition: The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. (The set of all images of elements of V.) For the matrix product case $A\vec{x}$, this is Col A.

Definition: The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in V such that $T(\mathbf{x}) = \mathbf{0}$. (The analog of the null space of a matrix.)

Theorem: Given linear transformation $T : V \longrightarrow W$, the range of T is a subspace of W and the kernel of T is a subspace of V.

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function y must satisfy if y is in the kernel of T.

To be in the kernel, $T(y) = 0$. But

$T(y) = \frac{dy}{dx} + \alpha y$ so y would satisfy

$$\frac{dy}{dx} + \alpha y = 0$$

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(b) Show that for any scalar c, $y = ce^{-\alpha x}$ is in the kernel of T.

For $y = ce^{-\alpha x}$, $\frac{dy}{dx} = -\alpha c e^{-\alpha x}$. Then

$$\frac{dy}{dx} + \alpha y = -\alpha c e^{-\alpha x} + \alpha c e^{-\alpha x} = 0 \text{ for all } x.$$

So y is in the kernel of T.

Section 4.3: Linearly Independent Sets and Bases

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.
The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero). If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation.**

Theorem: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ for $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example

Determine if the set is linearly dependent or independent in $\mathbb{P}_2$.

(a) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = 2t$, $\mathbf{p}_3 = t - 3$.

Note $\vec{p}_3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1 = \frac{1}{2}(2t) - 3(1) = t - 3.$

The set is linearly dependent.

(b) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

Consider $\quad c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0} = 0 + 0t + 0t^2$

$$2c_1 + c_2 t - c_3 t^2 = 0 + 0t + 0t^2.$$

This requires $\quad c_1 = c_2 = c_3 = 0$.

The set is linearly independent.

Example

Show that every vector $\mathbf{p} = p_0 + p_1 t + p_2 t^2$ in $\mathbb{P}_2$ can be written as a linear combination of $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$[1] where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

Letting $a_0 = \frac{1}{2}p_0$, $a_1 = p_1$ and $a_2 = -p_2$, then

$$a_0\vec{p}_1 + a_1\vec{p}_2 + a_2\vec{p}_3 = \frac{1}{2}p_0\cdot(2) + p_1 t + (-p_2)(-t^2)$$

$$= p_0 + p_1 t + p_2 t^2.$$

So $\vec{p}$ is in $\mathrm{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$.

[1] i.e. this set *spans* $\mathbb{P}_2$

Definition (Basis)

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Example

If A is an invertible $n \times n$ matrix, then we know[2] that (1) the columns are linearly independent, and (2) the columns span $\mathbb{R}^n$. Use this to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

We can form a 3x3 matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$.

We can use det(A) to check if this set is a basis.

[2] from our large theorem on invertible matrices from section 2.3

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}.$$

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= 3\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} + 0\cdot\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + (-6)\begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3(5-7) - 6(-4+2) = 6 \neq 0.$$

So the columns are lin. independent and span $\mathbb{R}^3$,

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$

Other Vector Spaces

Show that $\{1, t, t^2, t^3\}$ is a basis for $\mathbb{P}_3$[3].

If $\vec{p}$ is in $\mathbb{P}_3$, $\vec{p} = p_0 + p_1 t + p_2 t^2 + p_3 t^3$. $\vec{p}$ is a linear combo of $\{1, t, t^2, t^3\}$. Considering

$$c_1 + c_2 t + c_3 t^2 + c_4 t^3 = \vec{0} = 0 + 0t + 0t^2 + 0t^3,$$

requires $c_i = 0$ for $i = 1, \ldots, 4$. It is a linearly independent spanning set, hence a basis.

[3]The set $\{1, t, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$

Other Vector Spaces

Show that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M^{2\times 2}$.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $M^{2\times 2}$.

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The set spans $M^{2\times 2}$.

If $\quad c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$

then $$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This requires $c_i = 0$, $i = 1, \ldots, 4$. The set is linearly independent. The set is a basis of $M^{2\times 2}$.

A Spanning Set Theorem

Example: Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space V, and suppose that
(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.
Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

For $\vec{u}$ in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

Then $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2)$

$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2 = d_1\vec{v}_1 + d_2\vec{v}_2$.

where $d_1 = c_1 + c_3$, $d_2 = c_2 - 2c_3$. Hence $\vec{u}$ is in Span$\{\vec{v}_1, \vec{v}_2\}$.

Theorem:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

(a.) If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{array}{c} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{array} \end{array}.$$

Note $\vec{b}_2 = 4\vec{b}_1$, $\vec{b}_4 = 2\vec{b}_1 - \vec{b}_3$

$\vec{b}_1, \vec{b}_3, \vec{b}_5$ are linearly independent.

So a basis for Col B is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$