October 5 MATH 1113 sec. 51 Fall 2018

Section 5.1: Inverse Relations and Inverse Functions

Definition: Let S be a relation with domain D and range R. The inverse relation S^{-1} is the relation having domain R and range D defined by

$$(x, y) \in S^{-1} \quad \text{provided} \quad (y, x) \in S.$$

Definition: A function f is **one to one** if different inputs have different outputs. That is, f is one to one provided

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

Equivalently, f is a one to one function provided

$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

Inverse Function

Theorem: If f is a one to one function with domain D and range R, then its inverse f^{-1} is a function with domain R and range D. Moreover, the inverse function is defined by

$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x.$$

Characteristic Compositions:
If f is a one to one function with domain D, range R, and with inverse function f^{-1}, then

- for each x in D, $(f^{-1} \circ f)(x) = x$, and
- for each x in R, $(f \circ f^{-1})(x) = x$.

Example: $f(x) = \sqrt[3]{x-1}$

f is a one to one function. Verify that $f^{-1}(x) = x^3 + 1$ by showing that $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$.

$$(f^{-1} \circ f)(x) = f^{-1}(f(x))$$

$$= f^{-1}(\sqrt[3]{x-1})$$

$$= (\sqrt[3]{x-1})^3 + 1$$

$$= x - 1 + 1$$

$$= x$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x))$$

$$= f(x^3+1)$$

$f(x) = \sqrt[3]{x-1}$

$$= \sqrt[3]{(x^3+1)-1}$$

$$= \sqrt[3]{x^3+1-1}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

Question

The function $f(x) = \dfrac{1}{x+1}$ is one to one. Which of the following is its inverse function? (Hint: Check compositions $(f^{-1} \circ f)(x)$.)

(a) $f^{-1}(x) = x + 1$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x+1}\right)$$

(b) $f^{-1}(x) = \dfrac{1-x}{x}$

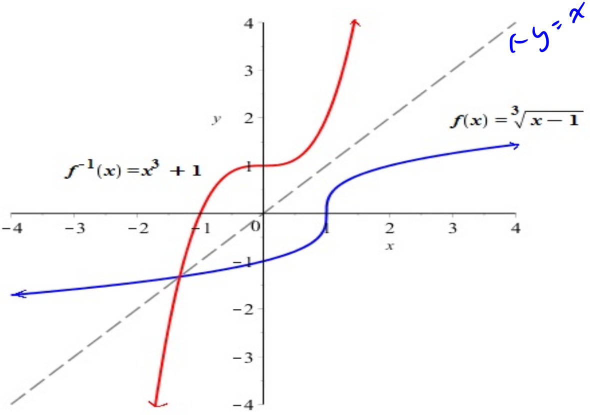
$$= \frac{1 - \frac{1}{x+1}}{\frac{1}{x+1}} \left(\frac{x+1}{x+1}\right)$$

(c) $f^{-1}(x) = \dfrac{x}{x+1}$

$$= \frac{x+1-1}{1} = \frac{x}{1} = x$$

Graphs

If (a, b) is a point on the graph of a function, then (b, a) is a point on the graph of its inverse. So the graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



Horizontal Line Test

The graph of a function must pass the vertical line test. We can ask what sort of curve would result in a vertical line upon being reflected in the line $y = x$.

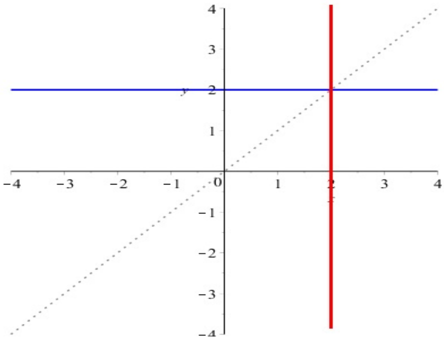


Figure: When a horizontal line is reflected in $y = x$, the result is a vertical line. So if two points of a graph are on one horizontal line, those points will be on the same vertical line when reflected.

Horizontal Line Test

Theorem: The function f is one to one if and only if its graph $y = f(x)$ interescts every horizontal line at most one times.

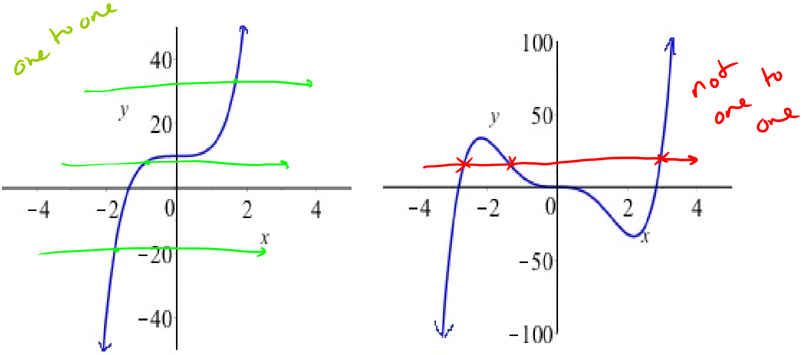
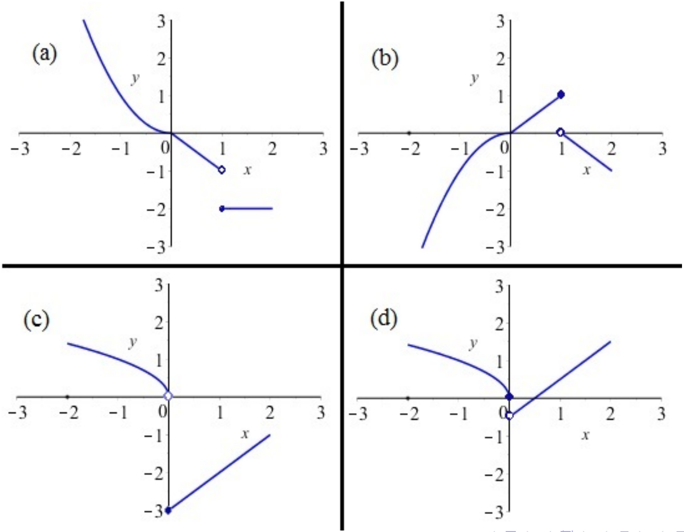


Figure: Left: A one to one function. Right: A function that is not one to one.

Question

Which plot shows a function whose inverse relation is a function?



Identifying an Inverse Function Formula

Given a one to one function $f(x)$, we can find[1] a formula for its inverse f^{-1} by the following steps

(1) Write $y = f(x)$.

(2) Interchange the variable names x and y.

(3) Solve for y using any necessary algebra.

(4) Replace y with $f^{-1}(x)$.

[1] It may be that the algebra becomes intractable. We may not have a *nice* formula for the inverse.

Example

Find f^{-1} given $f(x) = \dfrac{2x+1}{x+3}$. Verify that $(f^{-1} \circ f)(x) = x$.

$$y = \frac{2x+1}{x+3}$$ swap names $x \leftrightarrow y$

$$x = \frac{2y+1}{y+3}$$ solve for y

$$x(y+3) = 2y+1$$

$$xy + 3x = 2y + 1$$

$$xy - 2y = 1 - 3x$$

$$(x-2)y = 1-3x$$

$$y = \frac{1-3x}{x-2}$$

This "y" defines f^{-1}

$$f^{-1}(x) = \frac{1-3x}{x-2}$$

$$f(x) = \frac{2x+1}{x+3}$$

Check $(f^{-1} \circ f)(x)$

$$= f^{-1}(f(x))$$

$$= f^{-1}\left(\frac{2x+1}{x+3}\right)$$

$$= \frac{1 - 3\left(\frac{2x+1}{x+3}\right)}{\left(\frac{2x+1}{x+3}\right) - 2} \cdot \left(\frac{x+3}{x+3}\right)$$

$$= \frac{x+3-3(2x+1)}{2x+1-2(x+3)}$$

$$= \frac{x+3-6x-3}{2x+1-2x-6}$$

$$= \frac{-5x}{-5}$$

$$= x$$

as expected!

Restricting the Domain

Sometimes we wish to define an inverse function when a given function is not, in general, one to one. This can be done by restricting the domain of the original function.

The classic example would be the square root. Note $f(x) = x^2$ is not one to one if its domain is $(-\infty, \infty)$. However, if we consider the function $F(x) = x^2$ for $0 \leq x < \infty$, this function is one to one with inverse $F^{-1}(x) = \sqrt{x}$.

Restricting the Domain

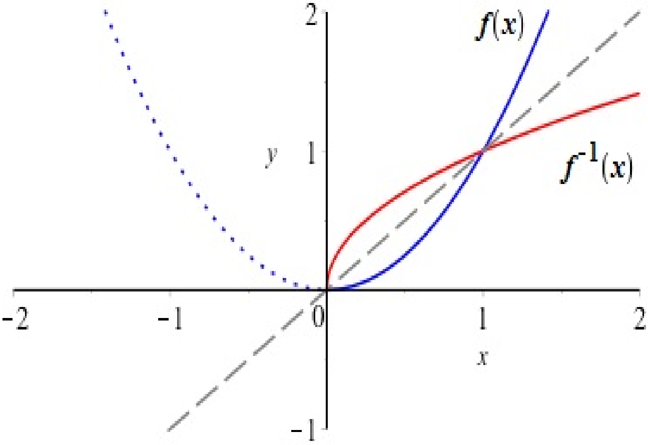


Figure: If the domain of $y = x^2$ is restricted to $[0, \infty)$, the graph passes the horizontal line test.