Oct. 5 Math 1190 sec. 51 Fall 2016

Section 3.3: Derivatives of Logarithmic Functions

Properties of logarithms can be used to simplify expressions characterized by products, quotients and powers.

Illustrative Example: Evaluate $\dfrac{d}{dx}\ln\left(\dfrac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right)$

By the chain rule, $\frac{d}{dx}\ln(f(x)) = \dfrac{f'(x)}{f(x)}$

Let's use properties of logs to break up our $\ln(f(x))$.

$$\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \ln\left(x^2\cos(2x)\right) - \ln\left(\sqrt[3]{x^2+x}\right)$$

$$= \ln x^2 + \ln \cos(2x) - \ln (x^2+x)^{1/3}$$

$$= 2\ln x + \ln \cos(2x) - \frac{1}{3}\ln (x^2+x)$$

Now we compute the derivative.

$$\frac{d}{dx} \ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \frac{d}{dx}\left(2\ln x + \ln \cos(2x) - \frac{1}{3}\ln(x^2+x)\right)$$

$$= 2\,\frac{1}{x} + \frac{-\sin(2x)\cdot 2}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

$$= \frac{2}{x} - \frac{2\sin(2x)}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

$$= \frac{2}{x} - 2\tan(2x) - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

Question

Evaluate the derivative. Use properties of logs to simplify the process.

$$\frac{d}{dx} \ln\left(\frac{\sqrt{x}}{\tan x + 1}\right)$$

Which expression is the correct derivative

(a) $\frac{1}{2x} - \frac{\sec^2 x}{\tan x}$

(b) $\frac{1}{2x} - \frac{\sec^2 x}{\tan x + 1}$

(c) $\frac{x}{2} - \tan x - 1$

(d) $\frac{1}{\frac{1}{2}x^{-1/2}} - \frac{1}{\sec^2 x}$

$$\ln\left(\frac{\sqrt{x}}{\tan x+1}\right) = \ln\sqrt{x} - \ln(\tan x+1)$$

$$= \frac{1}{2}\ln x - \ln(\tan x+1)$$

$$\frac{d}{dx}\ln\left(\frac{\sqrt{x}}{\tan x+1}\right) = \frac{1}{2}\,\frac{1}{x} - \frac{\sec^2 x}{\tan x+1}$$

Logarithmic Differentiation

We can use properties of logarithms to simplify the process of taking derivatives of expressions that are complicated by

products quotients and powers.

Illustrative Example: Evaluate $\dfrac{d}{dx}\left(\dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}\right)$

we'll introduce a log. If y is some differentiable function of x, then

$$\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx}$$ so that

$$\frac{dy}{dx} = y\left(\frac{d}{dx}\ln y\right)$$

This is useful if computing $\frac{d}{dx} \ln y$ is easier than computing $\frac{dy}{dx}$ directly.

Let $y = \dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}$ then

$$\ln y = \ln\left(\frac{x^2\sqrt{x+1}}{\cos^4(3x)}\right)$$

$$= \ln\left(x^2\sqrt{x+1}\right) - \ln \cos^4(3x)$$

$$= \ln x^2 + \ln (x+1)^{1/2} - \ln\left(\cos(3x)\right)^4$$

$$= 2\ln x + \frac{1}{2}\ln(x+1) - 4\ln\left(\cos(3x)\right)$$

So

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(2\ln x + \frac{1}{2}\ln(x+1) - 4\ln\left(\cos(3x)\right)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x} + \frac{1}{2}\frac{1}{x+1} - 4\frac{-\sin(3x)\cdot 3}{\cos(3x)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2(x+1)} + 12\frac{\sin(3x)}{\cos(3x)}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{2(x+1)} + 12\tan(3x)\right)$$

Using $y = \dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}$

$$\boxed{\text{we get} \quad \frac{dy}{dx} = \frac{x^2\sqrt{x+1}}{\cos^4(3x)}\left(\frac{2}{x} + \frac{1}{2(x+1)} + 12\tan(3x)\right)}$$

This process is called logarithmic differentiation.

Logarithmic Differentiation

If the differentiable function $y = f(x)$ consists of complicated products, quotients, and powers:

(i) Take the logarithm of both sides, i.e. $\ln(y) = \ln(f(x))$. Then use properties of logs to express $\ln(f(x))$ as a sum/difference of simpler terms.

(ii) Take the derivative of each side, and use the fact that $\frac{d}{dx}\ln(y) = \frac{\frac{dy}{dx}}{y}$.

(iii) Solve for $\frac{dy}{dx}$ (i.e. multiply through by y), and replace y with $f(x)$ to express the derivative explicitly as a function of x.

Example

Find $\frac{dy}{dx}$.

$$y = \frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}$$

Take log of both sides.

$$\ln y = \ln\left(\frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\right)$$

$$= \ln x^3 + \ln(4x-1)^5 - \ln(x+5)^{1/4}$$

$$= 3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)$$

Now take the derivative

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 3\frac{1}{x} + 5\frac{4}{4x-1} - \frac{1}{4}\frac{1}{x+5}$$

Isolate $\frac{dy}{dx}$

$$\frac{dy}{dx} = y\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4(x+5)}\right)$$

Now sub $y = \dfrac{x^3(4x-1)^5}{\sqrt[4]{x+5}}$

$$\boxed{\frac{dy}{dx} = \frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4(x+5)}\right)}$$

For some functions, logarithmic differentiation is required. For example, consider

$$y = x^x$$

The base is variable, so it's not an exponential.
The power is variable, so it's not a power function.
We don't have a rule for this. We can compute its derivative using logarithmic differentiation.

$$y = x^x \implies \ln y = \ln x^x = x \ln x$$

Using the product rule

$$\frac{d}{dx} \ln y = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{d}{dx} x\right) \ln x + x \left(\frac{d}{dx} \ln x\right)$$

$$= 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\implies \frac{dy}{dx} = y(\ln x + 1) \implies \boxed{\frac{d}{dx} x^x = x^x (\ln x + 1)}$$

Questions

Find $\frac{dy}{dx}$.

$$y = \frac{(x+3)(x-4)^2}{\sqrt{x}(x+1)^3}$$

$$\ln\left(\frac{(x+3)(x-4)^2}{\sqrt{x}(x+1)^3}\right) = \ln\left((x+3)(x-4)^2\right) - \ln\left(\sqrt{x}(x+1)^3\right)$$

$$= \ln(x+3) + \ln(x-4)^2 - \left(\ln x^{1/2} + \ln(x+1)^3\right)$$

(a) $\frac{dy}{dx} = \left[\frac{(x+3)(x-4)^2}{\sqrt{x}(x+1)^3}\right]\left(\frac{1}{x+3} + \frac{2}{x-4} - \frac{1}{2x} - \frac{3}{x+1}\right)$

(b) $\frac{dy}{dx} = \left[\frac{(x+3)(x-4)^2}{\sqrt{x}(x+1)^3}\right]\left(\frac{1}{x+3} + \frac{1}{(x-4)^2} - \frac{1}{\sqrt{x}} - \frac{1}{(x+1)^3}\right)$

(c) $\frac{dy}{dx} = \frac{1}{x+3} + \frac{2}{x-4} - \frac{1}{2x} - \frac{3}{x+1}$

The number e

We have already defined e by the limit

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

An alternative definition of the number e is given by

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Evaluate the limit

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$

Let's let k be defined by $\frac{2}{n} = \frac{1}{k}$

This gives $2k = n$. If $n \to \infty$

then $k \to \infty$ since $k = \frac{n}{2}$.

Now

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{\overset{n}{2k}}$$

$k \to \infty$ if $n \to \infty$

$\frac{2}{n}$

now use

$$a^{bc} = \left(a^b\right)^c$$

$$= \lim_{k\to\infty} \left(\left(1+\frac{1}{k}\right)^{k} \right)^{2}$$

$$= (e)^{2}$$

$$= e^{2}$$

If

$$\lim_{x\to c} f(x) = L$$

then

$$\lim_{x\to c} (f(x))^{2} = L^{2}$$